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Limits of solutions to a nonlinear second-order ODE

Cristian Vladimirescu

Department of Mathematics, University of Craiova, 13 A.I. Cuza Str., Craiova RO 200585, Romania

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This paper is dedicated to the memory of Professor Cezar Avramescu.

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ABSTRACT

In this paper, the existence of solutions to the equation $\ddot{x} + 2f(t)\dot{x} + \beta(t)x + g(t, x) = 0$, $t \geq 0$, is discussed. Our approach allows us achieve extension to the case of the whole real line, for which the existence of homoclinic solutions having zero limit at $\pm\infty$ is deduced. The result is obtained through the method of the Lyapunov function and differential inequalities.

1. Introduction

Consider the nonlinear second-order ODE

$$\ddot{x} + 2f(t)\dot{x} + \beta(t)x + g(t, x) = 0, \quad t \in \mathbb{R}_+, \tag{1.1}$$

where $f, \ \beta : \mathbb{R}_+ \to \mathbb{R}$, and $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are three given functions; $\mathbb{R}_+ = [0, +\infty)$. We give sufficient conditions for which Eq. (1.1) admits at least one solution $x : \mathbb{R}_+ \to \mathbb{R}$ fulfilling a condition of the type

$$\widetilde{w}(t) \leq \beta_0 x^2(t) + (\dot{x}(t) + f(t)x(t))^2 \leq \widetilde{v}(t), \quad t \in \mathbb{R}_+, \tag{1.2}$$

where $\widetilde{v}, \widetilde{w} : \mathbb{R}_+ \to \mathbb{R}_+^*$ are two functions depending on f and β, $\beta_0 > 0$ is constant, and $\mathbb{R}_+^* := (0, +\infty)$. Next, we present sufficient conditions for which $v(+\infty) := \lim_{t\to+\infty} v(t) = 0$. This fact assures the existence of a solution x to Eq. (1.1), which is not identically zero, such that $x(+\infty) = \dot{x}(+\infty) = 0$.

Eq. (1.1) is a basic mathematical model for the representation of damped nonlinear oscillatory phenomena. The properties which could be interesting are the stability, the boundedness and the vanishing of solutions at $+\infty$, and these have been intensively studied in, for example, [1–14]. The asymptotic stability of the null solution to Eq. (1.1) is researched in [3,4] (in the case where $\beta(t) = 1$ and $g(t, x) = x$), in [10] (in the case where $\beta(t) = 1$) and in [11], by using ingenious transformations (introduced in [3,4]), differential inequalities, and fixed point theorems.

Here, we reconsider Eq. (1.1) under more general assumptions and prove an existence result for solutions not identically zero and vanishing at $+\infty$ (see Theorem 2.1 below). The left side of inequality (1.2) is used to prove that the solution found is not identically zero. To establish inequality (1.2), we will use the method of the Lyapunov function and differential inequalities.

E-mail address: vladimirescucris@yahoo.com.

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Less studied but nonetheless important is the asymptotic behavior of the solutions on the whole real axis $\mathbb{R}$. Recently, in [15–17,11,18] the stability, the boundedness of solutions and the vanishing of solutions at $\pm\infty$ are studied. In the present paper, our hypotheses are more general than the ones from [11,18] and allow us to achieve extension of Theorem 2.1 to the whole real axis (see Theorem 4.1 below, for which the existence of homoclinic solutions to Eq. (4.1) vanishing at $\pm\infty$ is deduced).

2. The main result

The following hypotheses will be required:

(i) $f \in C^1(\mathbb{R}_+)$ and $f(t) \geq 0$ for all $t \geq 0$.

(ii) $\int_0^{+\infty} f(t)dt = +\infty$.

(iii) There exist two constants $h, K \geq 0$ such that

$$|f'(t) + f^2(t)| \leq Kf(t), \quad \forall t \in [h, +\infty).$$

(iv) $\beta \in C(\mathbb{R}_+)$, $\int_0^{+\infty} |\beta(t) - \beta_0|dt < +\infty$, and

$$K < 2/\gamma,$$

where $\beta_0 > 0$ is constant and $\gamma = \max\{1; 1/\sqrt{\beta_0}\}$.

(v) $g \in C(\mathbb{R}_+ \times \mathbb{R})$ and g is locally Lipschitzian in x.

(vi) g satisfies the following estimate:

$$|g(t, x)| \leq f(t)o(|x|), \quad \forall t \in \mathbb{R}_+,$$

where "o" denotes the usual Landau symbol.

The main result of this paper is the following theorem:

Theorem 2.1. *If the assumptions* (i)–(vi) *are fulfilled, then Eq.* (1.1) *admits a solution x not identically zero and having* $x(+\infty) = \dot{x}(+\infty) = 0$.

3. Proof of Theorem 2.1

Define on $\mathbb{R}^2$ the Lyapunov function

$$V(z) = \beta_0 x^2 + y^2, \quad z = (x, y)^T \in \mathbb{R}^2.$$

If we use the transformation (as in [3])

$$y := \dot{x} + f(t)x$$

then Eq. (1.1) becomes

$$\dot{z} = F(t, z), \tag{3.1}$$

where

$$F(t, z) = \begin{pmatrix} y - f(t)x \\ (f'(t) + f^2(t) - \beta(t))x - f(t)y - g(t, x) \end{pmatrix}$$

and $z = (x, y)^T \in \mathbb{R}^2$.

The derivative $\dot{V}$ of V along system (3.1) (see [19], pp. 50, 99) is

$$\dot{V}(z) = (\operatorname{grad} V, F).$$

Therefore, we obtain

$$\dot{V}(z) = -2f(t)(\beta_0 x^2 + y^2) + 2xy[f'(t) + f^2(t) + \beta_0 - \beta(t)] - 2g(t, x)y. \tag{3.2}$$

Consider for $z = (x, y)^T \in \mathbb{R}^2$ the norm $\|z\| = \sqrt{\beta_0 x^2 + y^2}$.

Since $V : \mathbb{R}^2 \to \mathbb{R}_+$ is continuous, $V(0) = 0$, $\lim_{\|z\| \to +\infty} V(z) = +\infty$, and $V(\mathbb{R}^2)$ is connected, it follows that for every $r_0 > 0$, there exists $z_0 \in \mathbb{R}$, $z_0 \neq 0$, such that $V(z_0) = r_0$. Therefore, $\forall n \in \mathbb{N}^* := \{1, 2, \ldots\}$ one can consider $z_n = (x_n, y_n)^T$ the unique solution to system (3.1) for which

$$V(z_n(1/n)) = \exp\left[\int_0^{1/n} \upsilon(s)ds\right], \tag{3.3}$$