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Blowup and self-similar solutions for two-component drift–diffusion systems

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We discuss asymptotic properties of solutions of two-component parabolic drift–diffusion systems coupled through an elliptic equation in two space dimensions. In particular, conditions for finite time blowup versus the existence of forward self-similar solutions are studied.

1. Introduction

Drift–diffusion systems are widely used for the modeling of various phenomena in continuum mechanics and biology. For instance, the system of evolution equations describing the interaction of charged particles in the mean field approximation, known as the Nernst–Planck–Debye–Hückel system (see (8) below), is used in plasma physics, electrolyte theory as well as semiconductor modeling [1,2]. Two-component generalizations of the classical Keller–Segel system in chemotaxis theory (e.g. [3]) are used to describe two species interacting via a diffused sensitivity agent which can be either a chemoattractant or a chemorepellent for each of the populations. The interaction of massive particles of two kinds through the gravitational potential generated by themselves can also be modeled by such a mean field model; see (7) below [4,5]. Also, the system (9) below has been introduced in [6]; cf. [7,5].

A general class of multicomponent systems of many species with the densities u_i, $i = 1, \ldots, k$, interacting via several sensitivity agents of densities v_j, $j = 1, \ldots, \ell$, has been studied in [8]. These populations may also collaborate or compete with each other. Wolansky proposed the following system of parabolic equations:

$$\nu_i \frac{\partial}{\partial t} u_i = \Delta u_i - \sum_j \vartheta_{ij} \nabla \cdot (u_i \nabla v_j), \tag{1}$$

$$\sigma_j \frac{\partial}{\partial t} v_j = \Delta v_j - \alpha v_j + \sum_j \gamma_{ij} u_i + f_j, \tag{2}$$

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describing the diffusion of species, their interactions, and the production and diffusion of sensitivity agents. In the case of bounded domains, the system (1)–(2) is to be supplemented with appropriate boundary conditions (either Neumann or no-flux for u_i's, and either Neumann or free for v_j's). He showed that under an algebraic condition of the absence of conflicts between species, the system (1)–(2) considered in two space dimensions has a variational structure, and the steady states can be studied using generalizations of the Moser–Trudinger inequality; see [9,8].

We will consider two-component systems with $i = 1, 2$, interacting through one sensitivity agent, $j = 1$, which diffuses instantaneously: $\sigma_j = 0$, so (2) is an elliptic equation. Our analysis will be carried out in two space dimensions, in the whole space $\mathbb{R}^2$, which is critical for a balance of diffusion and drift properties. In particular, due to the scaling properties, blowup conditions are expected to be expressed in terms of critical masses (or charges), i.e. the L^1 norms of u_i. Moreover, the existence of integrable self-similar solutions can be proved in some range of masses. Thus, our object of study is the parabolic–elliptic system

$$\frac{\partial}{\partial t}u_1 = \nabla \cdot (\nabla u_1 + \vartheta_1 u_1 \nabla v), \tag{3}$$

$$\frac{\partial}{\partial t}u_2 = \nabla \cdot (\nabla u_2 + \vartheta_2 u_2 \nabla v), \tag{4}$$

$$-\Delta v = \gamma_1 u_1 + \gamma_2 u_2, \tag{5}$$

where $u_1, u_2 \geq 0$ are densities of two species, and the masses (or charges) are

$$\int_{\mathbb{R}^2} u_i \, \mathrm{d}x = M_i, \quad i = 1, 2. \tag{6}$$

Here, for simplicity, $\vartheta_i, \gamma_i \in \{-1, 1\}$ but we may consider more general cases $\vartheta_i, \gamma_i \in \mathbb{R} \setminus \{0\}$ in essentially the same way.

However, if some of ϑ_i, γ_i are allowed to vanish, the classical single-component Keller–Segel system [3,10,11] arises for

$$\vartheta_1 = -1, \qquad \vartheta_2 = 0, \qquad \gamma_1 = 1, \qquad \gamma_2 = 0$$

(note that the equation for u_2 decouples), while

$$\vartheta_1 = 1, \qquad \vartheta_2 = 0, \qquad \gamma_1 = 1, \qquad \gamma_2 = 0$$

corresponds to a model of a cloud of identically charged particles [1]. Here $\vartheta_i\gamma_i < 0$ means that the particles of the ith species attract each other, while $\vartheta_i\gamma_i > 0$ signifies their mutual repulsion. Finally, $(\vartheta_1\gamma_1)(\vartheta_2\gamma_2) < 0$ results in a conflict of interest; cf. [8].

A priori, there are sixteen possible choices of signs for ϑ_i, γ_i. Clearly, exchanging the variables u_1, u_2 results in equivalent systems. Similarly, changing simultaneously the sign of all the parameters $\vartheta_1, \vartheta_2, \gamma_1, \gamma_2$ also gives the equivalent systems, denoted below by $\sim$.

Proceeding in this way we obtain the following list of the sixteen quadruples $\langle \vartheta_1, \vartheta_2, \gamma_1, \gamma_2 \rangle$ corresponding to six genuinely different systems, four of them without conflicts of interest, called here

$$\text{gravitational: } \langle -1, -1, 1, 1 \rangle \sim \langle 1, 1, -1, -1 \rangle, \tag{7}$$

$$\text{electric: } \langle 1, -1, 1, -1 \rangle \sim \langle -1, 1, -1, 1 \rangle, \tag{8}$$

$$\text{K–O}: \ \langle -1, 1, 1, -1 \rangle \sim \langle 1, -1, -1, 1 \rangle, \tag{9}$$

$$\text{repulsive: } \langle -1, -1, -1, -1 \rangle \sim \langle 1, 1, 1, 1 \rangle, \tag{10}$$

and also two kinds of systems with conflicts of interest:

$$\text{mixed: } \langle -1, -1, 1, -1 \rangle \sim \langle -1, -1, -1, 1 \rangle \sim \langle 1, 1, -1, 1 \rangle \sim \langle 1, 1, 1, -1 \rangle, \tag{11}$$

$$\text{uniform: } \langle -1, 1, 1, 1 \rangle \sim \langle 1, -1, 1, 1 \rangle \sim \langle 1, -1, -1, -1 \rangle \sim \langle -1, 1, -1, -1 \rangle. \tag{12}$$

The names that we coined for them are related to the type of interaction potential generated by the different components in the system. The system (7) corresponds to particles of two different masses that attract each other through the Newtonian potential. The system (8) describes particles of opposite charges interacting through the Coulombic potential. The system (9) is similar to (8) but the potential generated by the particles acts in the opposite direction compared to (8). The system (10) corresponds to particles of two different kinds that repel each other. The systems (8), (9) are well known, and are called Debye–Hückel and Kurokiba–Ogawa systems, respectively. For short, we call them simply electric and K–O systems, respectively.

2. Blowup of solutions

In this section we formulate simple algebraic criteria in terms of either masses or charges M_1, M_1 which are sufficient for a finite time blowup of nonnegative solutions of the Cauchy problem for the systems without conflicts of interest, with prescribed initial values $u_1(\cdot, 0)$ and $u_2(\cdot, 0)$ that satisfy (6). Formally, mass conservation and positivity preserving properties