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A blow-up criterion of strong solutions to a viscous liquid–gas two-phase flow model with vacuum in 3D

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ABSTRACT

In this paper, we get a blow-up criterion for the strong solutions to the 3D viscous liquid–gas two-phase flow model with vacuum. More precisely, we prove that the bound of $L^1_tL^\infty_x$ norm of the deformation tensor of velocity gradient controls the possible breakdown of the strong solutions.

1. Introduction

In this paper, we consider a simplified viscous liquid–gas two-phase flow model in three space dimension:

$$\begin{cases} m_t + \operatorname{div}(mu) = 0, \\ n_t + \operatorname{div}(nu) = 0, \\ (mu)_t + \operatorname{div}(mu \otimes u) + \nabla P(m, n) = \mu \Delta u + (\mu + \lambda)\nabla \operatorname{div} u, \quad \text{in } \Omega \times (0, \infty), \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^3$. Here $m = \alpha_l \rho_l$ and $n = \alpha_g \rho_g$ denote the liquid mass and gas mass, respectively; μ and λ are viscosity constants, satisfying

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0, \tag{1.2}$$

which implies $\mu + \lambda \geq \frac{1}{3}\mu > 0$.

The unknown variables $\alpha_l, \alpha_g \in [0, 1]$ denote respectively the liquid and gas volume fractions, satisfying the fundamental relation: $\alpha_l + \alpha_g = 1$. Furthermore, the other unknown variables ρ_l and ρ_g denote respectively the liquid and gas densities, satisfying equations of state: $\rho_l = \rho_{l,0} + \frac{P - P_{l,0}}{a_l^2}$, $\rho_g = \frac{P}{a_g^2}$, where a_l and a_g are sonic speeds, respectively, in the liquid and gas, and $P_{l,0}$ and $\rho_{l,0}$ are the reference pressure and density given as constants; u denotes velocity of the liquid and gas; P is the common pressure for both phases, which satisfies

$$P(m, n) = C^0 \left(-b(m, n) + \sqrt{b(m, n)^2 + c(n)} \right), \tag{1.3}$$

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with $C^0=\frac{1}{2}a_l^2, k_0=\rho_{l,0}-\frac{P_{l,0}}{a_l^2}>0, a_0=(\frac{a_g}{a_l})^2$ and

$$b(m,n)=k_0-m-\left(\frac{a_g}{a_l}\right)^2 n=k_0-m-a_0n,$$

$$c(n)=4k_0\left(\frac{a_g}{a_l}\right)^2 n=4k_0a_0n.$$

For more information about the system (1.1), please refer to [1–3] and references therein.

The system (1.1) is supplemented with the initial conditions:

$$(m,n,u)|_{t=0}=(m_0,n_0,u_0), \tag{1.4}$$

and any one of the three types of boundary conditions:

(1) Cauchy problem:

$$\Omega=\mathbb{R}^3 \quad \text{and} \quad (\text{in some weak sense})\, m,\ n,\ u \text{ vanish at infinity}. \tag{1.5}$$

(2) Dirichlet problem: in this case, Ω is a smooth bounded domain in $\mathbb{R}^3$, and

$$u=0 \quad \text{on } \partial\Omega. \tag{1.6}$$

(3) Navier-slip boundary condition: in this case, Ω is a smooth bounded domain in $\mathbb{R}^3$, and

$$u\cdot n=0, \qquad \operatorname{curl} u\times n=0 \quad \text{on } \partial\Omega. \tag{1.7}$$

The investigation of model (1.1) has been studied by many people during the past decade. There are a great deal of literature on the large time existence and behavior of solution to this model or related model. Let us review some previous works about the viscous liquid–gas two-phase flow model. For the model (1.1) in 1D, when the liquid is incompressible and the gas is polytropic, i.e., $P(m,n)=C\rho_l^\gamma(\frac{n}{\rho_l-m})^\gamma$, Evje and Karlsen in [4] studied the existence and uniqueness of the global weak solution to the free boundary value problem with $\mu=\mu(m)=k_1\frac{m^\beta}{(\rho_l-m)^{\beta+1}}$, $\beta\in(0,\frac{1}{3})$, when the fluids are connected to vacuum state discontinuously. Yao and Zhu [5] extended the results in [4] to the case $\beta\in(0,1]$, and also obtained the asymptotic behavior and regularity of the solution. Evje, et al. in [6] also studied the model with $\mu=\mu(m,n)=k_2\frac{n^\beta}{(\rho_l-m)^{\beta+1}}$ ($\beta\in(0,\frac{1}{3})$) in a free boundary setting when the fluids were connected to the vacuum state continuously, and obtained the global existence of the weak solution. Also, for the case of connecting to vacuum state continuously, Yao and Zhu in [7] obtained the global existence of the unique weak solution. Specifically, when both of the two fluids are compressible, one can be referred to the Ref. [8]. While there are few results about the well-posedness of the solutions of the multidimensional model until recently. Yao, et al. in their joint work [3] obtained the existence of the global weak solution to the 2D model when the initial energy is small. And they [9] established a blow-up criterion in terms of the upper bound of the density for the strong solutions to the 2D model in a smooth bounded domain under the assumption that $\inf m>0$ and $\inf n>0$. Wen, Yao and Zhu in [10] got a unique local strong solutions to the 3D viscous liquid–gas two-phase flow model in a smooth domain with vacuum, and established a blow-up criterion for the strong solutions in terms of the upper bound of the density under the assumption $\frac{25}{3}\mu>\lambda$. Recently, Hao and Li in [11] have got the global existence and uniqueness of the strong solution for the initial data close to an equilibrium and the local in time existence and uniqueness of the solution with general initial data in the framework of Besov spaces. In this paper, without the additional assumptions of μ and λ except for (1.2), we try to establish a blow-up criterion in terms of the upper bound of $L_t^1L_x^\infty$ norm of the deformation tensor for strong solutions to the 3D viscous liquid–gas two-phase flow model with vacuum.

Before stating the main result, we explain the notations and conventions used throughout this paper. We denote

$$\int f dx=\int_\Omega f dx.$$

For $1\le r\le\infty$, we denote the standard homogeneous and inhomogeneous Sobolev spaces as follows:

$$\begin{cases} L^r=L^r(\Omega), \qquad D^{k,r}=\{u\in L^1_{loc}(\Omega):\|\nabla^k u\|_{L^r}<\infty\},\\ W^{k,r}=L^r\cap D^{k,r}, \qquad H^k=W^{k,2}, \qquad D^k=D^{k,2},\\ D_0^1=\{u\in L^6:\|\nabla u\|_{L^2}<\infty,\ \text{and (1.5) or (1.6) or (1.7) holds}\},\\ H_0^1=L^2\cap D_0^1, \qquad \|u\|_{D^{k,r}}=\|\nabla^k u\|_{L^r}. \end{cases}$$

Theorem 1.1 (*Local Existence*). *Let Ω be a smooth domain in $\mathbb{R}^3$ and $q\in(3,6]$. Assume that $m_0,\ n_0\in W^{1,q}\cap H^1\cap L^1$, $u_0\in D_0^1\cap D^2$, $0\le \underline{s}_0m_0\le n_0\le\bar{s}_0m_0$ in $\overline{\Omega}$, where $\underline{s}_0$ and $\bar{s}_0$ are positive constants. Assume in addition that the following compatibility condition is also valid:*

$$-\mu\Delta u_0-(\mu+\lambda)\nabla\operatorname{div} u_0+\nabla P(m_0,n_0)=\sqrt{m_0}g, \quad \text{for some } g\in L^2. \tag{1.8}$$