Nonlinear Analysis 75 (2012) 5357–5361

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Renormings and fixed point property in non-commutative L_1-spaces II: Affine mappings

Carlos A. Hernández-Linares, Maria A. Japón *

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Sevilla, Tarfia s/n, 41012, Sevilla, Spain

ARTICLE INFO

Article history:
Received 12 March 2012
Accepted 24 April 2012
Communicated by S. Ahmad

MSC:
46B03
46L51
47H09
47H10

Keywords:
Fixed point theory
Renorming theory
Nonexpansive mappings
Affine mappings
Von Neumann algebras
Non-commutative L_1-spaces

ABSTRACT

In this paper, we prove that every noncommutative L_1-space associated to a finite von Neumann algebra can be renormed to satisfy the fixed point property for nonexpansive affine mappings. Particular examples are $L_1(\mathcal{R})$, where $\mathcal{R}$ is the hyperfinite II_1 factor and the function spaces $L_1[0, 1]$ and $L_1(\mu)$ for any σ-finite measure space. This property does not hold for the usual $\|\cdot\|_1$ norm.

1. Introduction

The present paper is a consequence of the authors' work in [1,2] and Randrianantoanina's generalization of Komlos' theorem to the setting of non-commutative L_1-spaces [3]. A Banach space X is said to have the fixed point property (FPP) if every nonexpansive mapping defined from a closed convex bounded subset into itself has a fixed point. It is well-known that classical nonreflexive Banach spaces such as c_0, ℓ_1 and $L_1[0, 1]$ fail to have the FPP. Lin [4] showed that ℓ_1 can be renormed to satisfy the FPP. It is still unknown whether c_0 or $L_1[0, 1]$ can be renormed to have the FPP. In [1], the authors proved that there exist some closed subspaces of $L_1[0, 1]$ (non-isomorphic to ℓ_1) that can also be renormed to satisfy the FPP. These results were extended in the case of non-commutative L_1-spaces in [2].

Let C be a convex subset. A mapping $T : C \to C$ is said to be *affine* if $T(\lambda x+(1-\lambda)y) = \lambda Tx+(1-\lambda)Ty$ whenever $x, y \in C$ and $\lambda \in [0, 1]$. Fixed point theorems for affine mappings have been widely studied [5–9]. Moreover, affine mappings have been useful to characterize weak compactness in Banach spaces by means of fixed point theorems: if C is a convex bounded subset of a Banach space X, then C is weakly compact if and only if for every closed convex subset $K \subset C$ and for every affine continuous mapping $T : K \to K$, there exists a fixed point [5]. In fact, the continuity condition can be replaced by nonexpansiveness whenever $X = L_1[0, 1]$ or more generally whenever X is an L-embedded Banach space [5,6]. Notice that the affine condition in the previous characterization cannot be omitted [10].

A Banach space X is said to have the fixed point property for affine mappings (A-FPP) if every affine nonexpansive mapping defined from a closed convex bounded subset into itself has a fixed point. Since the classical nonreflexive sequence

* Corresponding author.
E-mail addresses: carloshl@us.es (C.A. Hernández-Linares), japon@us.es (M.A. Japón).

0362-546X/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2012.04.050

spaces ℓ_1 and c_0 fail to have the A-FPP (see for instance [11, Chapter 3]), every Banach space which contains an isometric copy of ℓ_1 or c_0 also fails this property. What is more, it can be checked that every Banach space which contains an asymptotically isometric copy of either ℓ_1 or c_0 also fails the A-FPP. This implies that every nonreflexive subspace of $L_1[0, 1]$ or more generally, every nonreflexive subspace of the noncommutative space $L_1(\mathcal{M})$ associated to a von Neumann algebra fails to have the A-FPP, because they contain asymptotically isometric copies of ℓ_1 [12].

In this paper, we consider a finite von Neumann algebra $\mathcal{M}$ and let $L_1(\mathcal{M})$ be the corresponding noncommutative L_1-space, which fails the A-FPP for the usual $\|\cdot\|_1$ norm defined by $\|x\|_1 = \tau(|x|)$ where τ is a normal finite faithful trace on $\mathcal{M}$. Our main purpose is to obtain an equivalent norm $|||\cdot|||$ in $L_1(\mathcal{M})$ such that $(L_1(\mathcal{M}), |||\cdot|||)$ does satisfy the A-FPP. Moreover, we will show that this renorming can be chosen as 'close' to the usual norm as we want. In the $L_1[0, 1]$ case, the renorming can be defined by using non-increasing rearrangement functions and maximal functions. In particular, we will deduce that the above characterization of weak compactness in subsets of $L_1[0, 1]$ by using fixed point theorems for affine nonexpansive mappings does not hold if we change the $\|\cdot\|_1$ norm for an equivalent one, independently of the Banach–Mazur distance between them.

2. Preliminaries

For background results concerning the fixed point property for nonexpansive mappings, the reader can consult [11] or [13] and the references therein.

As a consequence of the Schauder–Tychonoff Theorem (see [14, p. 74]), every reflexive Banach space satisfies the A-FPP. This is due to the fact that every affine continuous self-mapping defined on a closed convex set is weakly continuous and bounded closed subsets are weakly compact whenever X is reflexive. Following the proofs given in [15,16] regarding to the failure of the FPP (see also Theorems 2.3 and 2.4 in [13], Chapter 9), for nonreflexive Banach spaces we can state the following.

Theorem 2.1. *Let X be a Banach space which contains an asymptotically isometric copy of either ℓ_1 or c_0. Then X fails to have the A-FPP.*

In particular, we can deduce that every nonreflexive subspace of $L_1[0, 1]$ or more generally every nonreflexive subspace of $L_1(\mathcal{M})$ fails to have the A-FPP for its usual norm. Also that every nonreflexive subspace of an M-embedded Banach space (such as the space of the compact operators in a Hilbert space $K(H)$) fails to satisfy the A-FPP (see [12]).

For definition and examples of non-commutative L_1-spaces the reader can consult [2] and the references therein. We assume that $\mathcal{M}$ is a finite von Neumann algebra on a separable Hilbert space and τ a finite normal faithful trace on $\mathcal{M}$. We denote by $L_1(\mathcal{M})$ the corresponding non-commutative L_1-space with usual norm $\|x\|_1 = \tau(|x|)$.

Every commutative von Neumann algebra is finite. In the case of $L_\infty(\mu)$, for (Ω, Σ, μ) a σ-finite measure space, we can consider the following finite trace:

$$\tau(f) = \sum_{n=1}^{\infty} \frac{1}{2^n \mu(\Omega_n)} \int_{\Omega_n} f d\mu, \quad f \in L_\infty(\mu);$$

where $\Omega = \cup \Omega_n$, the collection $\{\Omega_n\}$ is pairwise disjoint and $\mu(\Omega_n) < +\infty$. Here $L_1(\mu)$ is isometric to $L_1(\nu)$ with

$$\nu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n \mu(\Omega_n)} \mu(A \cap \Omega_n).$$

Thus, we can treat σ-finite measure $L_1(\mu)$ spaces as if they were finite up to isometry. In fact, it is a classical result (see for instance [17, Chapter 7]) that every commutative von Neumann algebra $\mathcal{M}$ can be isometrically identified with $L_\infty(\Omega, \Sigma, \mu)$ for some abstract measure space (Ω, Σ, μ).

We recommend [18–22] as important references in the framework of von Neumann algebras.

3. An equivalent norm in $L_1(\mathcal{M})$ with the A-FPP

Following the same arguments as in the proof of Theorem 1 in [1], we can state the following.

Theorem 3.1. *Let X be a Banach space endowed with a linear topology $\mathcal{T}$ and a family of seminorms $\{R_k(\cdot)\}_{k\geq 1}$ which satisfy the following properties:*

(I) $R_1(x) = \|x\|$ *while for* $k \geq 2$, $R_k(x) \leq \|x\|$ *for all* $x \in X$.
(II) $\lim_k R_k(x) = 0$ *for all* $x \in X$.
(III) *If* $x_n \xrightarrow{\mathcal{T}} 0$ *is norm-bounded and* $k \geq 1$ *we have*

$$\limsup_n R_k(x_n) = \limsup_n \|x_n\|_1.$$