Nonlinear Analysis 75 (2012) 3723–3745

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Hölder estimate for non-uniform parabolic equations in highly heterogeneous media

Li-Ming Yeh

Department of Applied Mathematics, National Chiao Tung University, Hsinchu, 30050, Taiwan, ROC

ARTICLE INFO

Article history:
Received 7 October 2011
Accepted 26 January 2012
Communicated by S. Carl

MSC:
35K10
35K20
35K65
35K67

Keywords:
Highly heterogeneous media
Pseudo-differential operator
Paramatrix
Strict solution
Infinitesimal generator
Numerical range

ABSTRACT

Uniform bound for the solutions of non-uniform parabolic equations in highly heterogeneous media is concerned. The media considered are periodic and they consist of a connected high permeability sub-region and a disconnected matrix block subset with low permeability. Parabolic equations with diffusion depending on the permeability of the media have fast diffusion in the high permeability sub-region and slow diffusion in the low permeability subset, and they form non-uniform parabolic equations. Each medium is associated with a positive number ϵ, denoting the size ratio of matrix blocks to the whole domain of the medium. Let the permeability ratio of the matrix block subset to the connected high permeability sub-region be of the order $\epsilon^{2\tau}$ for $\tau \in (0, 1]$. It is proved that the Hölder norm of the solutions of the above non-uniform parabolic equations in the connected high permeability sub-region are bounded uniformly in ϵ. One example also shows that the Hölder norm of the solutions in the disconnected subset may not be bounded uniformly in ϵ.

1. Introduction

Uniform Hölder estimate for the solutions of non-uniform parabolic equations in highly heterogeneous media is presented. The equations have many applications in multi-phase flows in porous media, the stress in composite materials, and so on (see [1–4] and references therein). The media $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) contain a connected high permeability sub-region and a disconnected matrix block subset with low permeability. Let $\partial\Omega$ denote the boundary of Ω, $\epsilon \in (0, 1)$, $\Omega(2\epsilon) \equiv \{x \in \Omega |\ \mathrm{dist}(x, \partial\Omega) > 2\epsilon\}$, and $Y \equiv (0, 1)^n$ denote a cell consisting of a sub-domain Y_m completely surrounded by another connected sub-domain Y_f ($\equiv Y \setminus \overline{Y}_m$). The disconnected matrix block subset of Ω is $\Omega^\epsilon_m \equiv \{x |\ x \in \epsilon(Y_m + j) \subset \Omega(2\epsilon)$ for some $j \in \mathbb{Z}^n\}$ with boundary $\partial\Omega^\epsilon_m$, and the connected sub-region is $\Omega^\epsilon_f \equiv \Omega \setminus \overline{\Omega}^\epsilon_m$. The non-uniform parabolic equations (see [4]) in $[0, T] \times \Omega$ are

$$\begin{cases} \partial_t U_\epsilon - \nabla \cdot (\Lambda^\epsilon_\tau \nabla U_\epsilon) = F_\epsilon & \text{in } (0, T] \times \Omega, \\ U_\epsilon = 0 & \text{on } (0, T] \times \partial\Omega, \\ U_\epsilon = U_{\epsilon,0} & \text{in } \{0\} \times \Omega, \end{cases} \tag{1.1}$$

where $\tau \in (0, \infty)$, $\Lambda^\epsilon_\tau \equiv \begin{cases} \mathbf{K}_\epsilon & \text{in } \Omega^\epsilon_f \\ \epsilon^{2\tau}\mathbf{k}_\epsilon & \text{in } \Omega^\epsilon_m \end{cases}$ (depending on the permeability of Ω), and both $\mathbf{K}_\epsilon$, $\mathbf{k}_\epsilon$ are positive smooth functions in Ω.

E-mail address: liming@math.nctu.edu.tw.

0362-546X/$ – see front matter
doi:10.1016/j.na.2012.01.027

Since $\epsilon \in (0,1)$, equations in (1.1) are non-uniform parabolic equations with discontinuous coefficients. In [5], existence of solution in $W_p^{2,1}([0,T]\times\Omega)$ space for uniform parabolic equations with discontinuous coefficients can be found. For non-uniform parabolic equations with smooth coefficients, existence of solution in $C^{2,\alpha}([0,T]\times\Omega)$ space was studied in [6]. It is also known that if F_ϵ, $U_{\epsilon,0}$ are smooth, a piecewise regular solution of (1.1) exists uniquely for each ϵ and, by the energy method, the H^1 norm of the parabolic solution of (1.1) in the connected high permeability sub-region is bounded uniformly in ϵ [2,7]. Hölder continuity of the parabolic solution of (1.1) in $[0,T]\times\Omega$ is proved for each ϵ [7], but the Hölder norm of the solution may go to infinity as $\epsilon \searrow 0$. In [4], convergence of solution of (1.1) in $L^\infty([0,T];L^2(\Omega))$ space as $\epsilon\searrow 0$ was obtained. Many studies of the uniform estimate in ϵ for the solutions of the elliptic equations in heterogeneous media had been done [2,3,8–11], but not the case for parabolic equations. The existence of piecewise regular solutions for elliptic diffraction equations in Hilbert space was considered in [2,9]. The uniform Lipschitz estimate in ϵ for a Laplace equation in perforated domains was given in [11], and a uniform L^p estimate in ϵ of the same problem was considered in [10]. A Lipschitz estimate for uniform elliptic equations was studied in [3]. Uniform Hölder and Lipschitz estimates in ϵ for uniform elliptic equations in periodic domains were obtained in [8].

This work is to present a uniform Hölder estimate in ϵ for the solutions of the non-uniform parabolic equations with discontinuous coefficients. More precisely, the Hölder norm of the non-uniform parabolic solutions in the connected high permeability sub-region is shown to be bounded uniformly in ϵ. However, the Hölder norm of the solutions in the disconnected subset may not be bounded uniformly in ϵ. This is due to the non-zero source in the disconnected subset. In Section 2, we present one example to show that. Certainly this is different from usual uniform parabolic equation cases, in which solutions are regular in the whole time–space domains. From the proof, we can see that the results are established for complex-valued solutions. On the other hand, one also notes that a complex-valued solution of (1.1) with complex-valued coefficients may be discontinuous or even unbounded [12]. A similar case could be found in elliptic equations with complex-valued coefficients (see [13]). It seems that the techniques used here could be used to study more general systems of elliptic type and parabolic type, and this will be pursued later. Some related uniform regularity results in the case of elliptic systems can be seen in [14,15].

The rest of the work is organized as follows: Notation and main results are stated in Section 2. The main results are proved in Section 3 based on semigroup theory and on uniform Hölder estimate in ϵ for non-uniform elliptic equations. To apply semigroup theory, an infinitesimal generator of an analytic semigroup from elliptic equations is required. So a $W^{2,p}$ estimate for solutions of elliptic diffraction equations is derived in Section 4. Two convergence results for solutions of non-uniform elliptic equations are shown in Section 5. By results in Section 5, a uniform Hölder estimate in ϵ for non-uniform elliptic solutions is proved in Section 6.

2. Notation and main result

Let $\overline{\Omega}$ be the closure of the domain Ω. Let $L^p(\Omega)$ (resp. $H^k(\Omega)$, $W^{k,p}(\Omega)$) denote a complex Sobolev space with norm $\|\cdot\|_{L^p(\Omega)}$ (resp. $\|\cdot\|_{H^k(\Omega)}$, $\|\cdot\|_{W^{k,p}(\Omega)}$), $W_0^{1,p}(\Omega) \equiv \{\varphi \in W^{1,p}(\Omega) | \varphi|_{\partial\Omega} = 0\}$, $H_0^1(\Omega) \equiv W_0^{1,2}(\Omega)$, $C_0^\infty(\Omega)$ be the set containing all infinite differentiable functions with compact support in Ω, $C(\overline{\Omega})$ consist of all continuous functions in $\overline{\Omega}$ with norm $\|\cdot\|_{C(\overline{\Omega})}$, $C^\sigma(\overline{\Omega})$ (resp. $C^{1,\sigma}(\overline{\Omega})$) denote a Hölder space with norm $\|\cdot\|_{C^\sigma(\overline{\Omega})}$ (resp. $\|\cdot\|_{C^{1,\sigma}(\overline{\Omega})}$), and $[\varphi]_{C^\sigma(\overline{\Omega})}$ (resp. $[\varphi]_{C^{1,\sigma}(\overline{\Omega})}$) denote the Hölder semi-norm of φ (resp. $\nabla\varphi$) for $k \ge -1$, $p \in [1,\infty]$, and $\sigma \in (0,1]$ [16,17]. If φ is a complex function, $\overline{\varphi}$ denotes its complex conjugate. If $\mathbf{B}_1$ and $\mathbf{B}_2$ are two Banach spaces, $\mathcal{L}(\mathbf{B}_1,\mathbf{B}_2)$ is the set of all bounded linear maps from $\mathbf{B}_1$ to $\mathbf{B}_2$ with norm $\|\cdot\|_{\mathcal{L}(\mathbf{B}_1,\mathbf{B}_2)}$. For any Banach space $\mathbf{B}$, define $\|\varphi_1,\varphi_2,\ldots,\varphi_m\|_{\mathbf{B}} \equiv \|\varphi_1\|_{\mathbf{B}} + \|\varphi_2\|_{\mathbf{B}} + \cdots + \|\varphi_m\|_{\mathbf{B}}$, denote its dual space by $\mathbf{B}'$, and denote the pairing between $\mathbf{B}$ and its dual space $\mathbf{B}'$ by $\langle\cdot,\cdot\rangle_{\mathbf{B},\mathbf{B}'}$. $L^\infty(I;\mathbf{B}) \equiv \{\varphi : I \to \mathbf{B} | \sup_{t\in I}\|\varphi(t)\|_{\mathbf{B}} < \infty\}$. The function spaces $C(I;\mathbf{B})$, $C^\sigma(I;\mathbf{B})$ for $\sigma\in(0,1]$ and an interval $I\subset\mathbb{R}$ are defined as those in pages 1, 3 [18]. $B_r(x)$ represents a ball centered at x with radius r. For any domain $\mathbb{D}$, $\overline{\mathbb{D}}$ is the closure of $\mathbb{D}$, $\partial\mathbb{D}$ is the boundary of $\mathbb{D}$, $\mathbb{D}/r \equiv \{x | rx \in \mathbb{D}\}$, $|\mathbb{D}|$ is the volume of $\mathbb{D}$, and $\mathcal{X}_{\mathbb{D}}$ is the characteristic function on $\mathbb{D}$. For any $\varphi \in L^1(B_r(x)\cap\Omega)$,

$$(\varphi)_{x,r} \equiv \fint_{B_r(x)\cap\Omega} \varphi(y)dy \equiv \frac{1}{|B_r(x)\cap\Omega|}\int_{B_r(x)\cap\Omega}\varphi(y)dy.$$

For any $p\in(1,\infty)$, $\tau\in(0,\infty)$, and $\epsilon\in(0,1)$,

$$\begin{cases} \mathcal{A}_\tau^\epsilon\varphi \equiv -\nabla\cdot(\Lambda_\tau^\epsilon\nabla\varphi), \\ \mathbb{B}_p(\mathcal{A}_\tau^\epsilon) \equiv \left\{\varphi\in W_0^{1,p}(\Omega) |\, \varphi\in W^{2,p}(\Omega_f^\epsilon)\cap W^{2,p}(\Omega_m^\epsilon), \mathbf{K}_\epsilon\nabla\varphi\cdot\vec{\mathbf{n}}^\epsilon|_{\partial\Omega_m^\epsilon} = \epsilon^{2\tau}\mathbf{k}_\epsilon\nabla\varphi\cdot\vec{\mathbf{n}}^\epsilon|_{\partial\Omega_m^\epsilon}\right\}, \end{cases}$$

where $\vec{\mathbf{n}}^\epsilon$ is a normal vector on $\partial\Omega_m^\epsilon$. It is not difficult to see that $\mathbb{B}_p(\mathcal{A}_\tau^\epsilon)$ with norm $\|\varphi\|_{\mathbb{B}_p(\mathcal{A}_\tau^\epsilon)} \equiv \|\mathcal{A}_\tau^\epsilon\varphi\|_{L^p(\Omega)}$ is a normed space. Let $\overline{\mathbb{B}_p(\mathcal{A}_\tau^\epsilon)}$ denote the closure of $\mathbb{B}_p(\mathcal{A}_\tau^\epsilon)$ in L^p space (we shall see $\overline{\mathbb{B}_p(\mathcal{A}_\tau^\epsilon)} = L^p(\Omega)$ from Lemma 3.4). For any $\lambda,\nu>0$, we define

$$\hat{\mathbf{K}}_{\lambda,\nu}(x) \equiv \mathbf{K}_\lambda(\nu x) \quad \text{and} \quad \hat{\mathbf{k}}_{\lambda,\nu}(x) \equiv \mathbf{k}_\lambda(\nu x). \tag{2.1}$$

Let $Y_m \subset \mathbf{D} \subset Y = Y_f \cup \overline{Y}_m$ satisfy

$$\min\{\mathrm{dist}(Y_m,\partial\mathbf{D}), \mathrm{dist}(\mathbf{D},\partial Y)\} > 0. \tag{2.2}$$