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The spectrum of the p-Laplacian with singular weight

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We use the Hardy–Sobolev inequality to characterize the first eigenvalue λ_1 of the p-Laplacian with singular weight. In some cases it is shown that λ_1 is positive simple, isolated and has a nonnegative corresponding eigenfunction ϕ_1. Higher eigenvalues, in particular the second one, are also determined.

1. Introduction

In the present paper we study the weighted eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain, $1 < p < \infty$ and $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$. This problem was addressed in [1–9] for domains with boundary at least C^2 and bounded weight. These works proved that there exists a first eigenvalue $\lambda_1 > 0$ (see (6) and (7)), which is simple in the sense that two eigenfunctions corresponding to it are proportional. Moreover, the corresponding first eigenfunction ϕ_1 can be assumed to be positive. Here we will assume that $\partial\Omega$ is a piecewise C^1 domain. The simplicity of λ_1 with $m \equiv 1$ without any regularity assumption on the boundary was established in [10] with the aid of special test functions. We follow this procedure here. In [1] the isolation from the left hand side of λ_1 was further proved. With that aim, a nodal set estimate for eigenfunctions corresponding to eigenvalues greater than λ_1 was obtained. We also prove such an estimate in the present paper.

Our main tool is the Hardy–Sobolev inequality proved in [11]. The optimal constant problem for such an inequality is addressed in [12]. The relationship between the Hardy–Sobolev inequality and the spectrum of a class of nonlinear differential operators has been addressed in [13]; see also [14] for one-dimensional results dealing with singular eigenvalue problems. An inspection of the proof in [11] reveals that the Hardy–Sobolev inequality holds true in a domain with piecewise C^1 boundary, on choosing local charts avoiding points where $\partial\Omega$ is nondifferentiable.

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Lemma 1.1. *If $\partial\Omega$ is piecewise C^1, then*

$$\left\| \frac{u}{\delta^\tau} \right\|_{L^t(\Omega)} \le C \|\nabla u\|_{L^q(\Omega)} \quad \text{for every } u \in W_0^{1,q}(\Omega)$$

for $\frac{1}{t} = \frac{1}{q} - \frac{1-\tau}{N}$ if $q < N$ and for $\frac{1}{t} = \frac{\tau}{q}$ if $q \ge N$, where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$, $\tau \in [0, 1]$ and $C = C(N, q, \tau) > 0$. In the case $t = q = p$, no regularity on $\partial\Omega$ is needed.

The weight m may be unbounded and change sign. We assume that $m\delta^\tau \in L^a(\Omega)$ and $m^+ \not\equiv 0$, where a, τ and p satisfy one of the following conditions:

$$\partial\Omega \text{ is piecewise } C^1, \quad 0 < \tau < 1, \qquad \frac{p}{1-\tau} \le a \quad \text{and} \quad a \le \frac{Np}{N - \tau p} \quad \text{if } N > \tau p; \tag{2}$$

$$\partial\Omega \text{ is piecewise } C^1, \quad 0 < \tau < 1, \quad \text{and} \quad p < \frac{N}{1-\tau} < a; \tag{3}$$

$$\partial\Omega \text{ is piecewise } C^1, \quad \tau = 1 \quad \text{and} \quad a = \infty; \tag{4}$$

$$\Omega \text{ is any bounded domain}, \quad \tau = 0 \quad \text{and} \quad a = \infty. \tag{5}$$

Condition (5) implies $m\delta^\tau = m \in L^\infty(\Omega)$, including results of the previously cited papers; see also Example 1.2 below. Here $\partial\Omega$ is piecewise C^1 except for (5). Conditions (2), (3), (4) or (5) are enough to set up a variational framework, namely the constrained minimization

$$\lambda_1 = \inf\{A(u) : u \in W_0^{1,p}(\Omega) \text{ and } B(u) = 1\}, \tag{6}$$

where $A, B : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ are the C^1 functions defined by

$$A(u) = \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} \quad \text{and} \quad B(u) = \frac{1}{p}\int_\Omega m(x)|u|^p dx. \tag{7}$$

Example 1.2. We will exhibit now a weight m such that $m\delta^\tau \in L^a(\Omega)$ with $m^+ \not\equiv 0$, where a, τ and p satisfy (3). The weight $m(x) = \delta(x)^{-\beta} = (1-|x|)^{-\beta}$ is admissible in the open unit ball of $\mathbb{R}^N$, i.e., $\Omega = B_1(0)$. We need to choose $\beta > 0$ adequately. We fix $p < N$, say, $p = 3/2$ and $N = 3$. We also choose $\tau = 1/2$ and $b = 7/4$; then $a = 21/2$. Thus, for $1/2 < \beta < 25/42$, we conclude that $m \notin L^{N/p}(\Omega) = L^2(\Omega)$, but $m\delta^{1/2} \in L^{21/2}(\Omega)$. More generally, given N and $1 < p < N$, it is possible to choose τ, b and β such that $m \notin L^{N/p}(\Omega)$, but $m\delta^\tau \in L^a(\Omega)$.

Example 1.2 is not included in the eigenvalue papers quoted above, since m is not bounded. This example does not verify the assumptions of [15], since it is needed there that $m \in L^s(\Omega)$, if $s > N/p$ and $1 < p \le N$. In [16,17] it is allowed that $m \in L^{N/p}(\Omega)$ if $1 < p \le N$. In [17] they only show that $\phi_1 \ge 0$ (no strict positiveness). In [16] they show simplicity and uniqueness of ϕ_1, but no isolation of λ_1 and λ_2 is addressed.

Section 2 is devoted to the study of λ_1 and its corresponding eigenfunction ϕ_1. We prove the following results.

Theorem 1.3. *If one assumes that $\partial\Omega$ is piecewise C^1 and that $m\delta^\tau \in L^a(\Omega)$ with $m^+ \not\equiv 0$, where a, τ and p satisfy (2), (3), (4) or (5), then the number λ_1 is attained by some $\phi_1 \in W_0^{1,p}(\Omega)$, where we may assume that $\phi_1 \ge 0$ a.e. in Ω, $\phi_1^+ \not\equiv 0$. Moreover, λ_1 is positive and isolated.*

Proposition 1.4. *If one assumes the same conditions as for Theorem 1.3, for an eigenfunction v corresponding to an eigenvalue $\lambda > \lambda_1$ there is a constant C independent of v such that $|\{x \in \Omega : v(x) < 0 \text{ a.e. in } \Omega\}| \ge C$.*

In some steps of our proofs we will need to use the Harnack inequality. With that aim, according to [18, Section 5] we make the following definitions involving locally integrable weights. Let $\varepsilon(\rho)$ be a smooth function defined for $\rho > 0$ such that

$$\varepsilon(\rho) \to 0 \quad \text{as } \rho \to 0^+ \quad \text{and} \quad \int_0^{\rho*} \frac{\varepsilon(\rho)}{\rho} d\rho < \infty \tag{8}$$

for some $\rho^* > 0$. We denote by $K_{x_0}(\rho)$ an N-dimensional cube contained in Ω whose edges are of length ρ and are parallel to the coordinate axes. We define

$$L^t_{\varepsilon(\rho)}(\Omega) = \{u \in L^t(\Omega) : \|u\|_{t,\varepsilon(\rho),\Omega} < \infty\}, \tag{9}$$

where

$$\|u\|_{t,\varepsilon(\rho),\Omega} = \sup\left\{ \frac{\|u\|_{L^t(K_{x_0}(\rho)\cap\Omega)}}{\varepsilon(\rho)} : x_0 \in \Omega, \rho > 0 \right\}. \tag{10}$$