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Four types of nonlinear scalarizations and some applications in set optimization

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ABSTRACT

There are two types of criteria of solutions for the set-valued optimization problem, the vectorial criterion and set optimization criterion. The first criterion consists of looking for efficient points of set valued map and is called set-valued vector optimization problem. On the other hand, Kuroiwa–Tanaka–Ha started developing a new approach to set-valued optimization which is based on comparison among values of the set-valued map. In this paper, we treat the second type criterion and call set optimization problem. The aim of this paper is to investigate four types of nonlinear scalarizing functions for set valued maps and their relationships. These scalarizing functions are generalization of Tammer–Weidner's scalarizing functions for vectors. As applications of the scalarizing functions for sets, we present nonconvex separation type theorems, Gordan's type alternative theorems for set-valued map, optimality conditions for set optimization problem and Takahashi's minimization theorems for set-valued map.

1. Introduction

Luc [1] and Gerth(Tammer)–Weidner [2] introduced sublinear scalarizing functions for vectors. They also gave nonconvex separation theorems and these functions have wide applications in vector optimization problems.

Let Y be a topological vector space ordered by a closed convex cone $C \subset Y$. Let X be a nonempty set and $F : X \to 2^Y$ a set-valued map with domain X ($F(x) \neq \emptyset$ for each $x \in X$). The set-valued optimization problem is formalized as follows:

$$(\mathrm{P}) \quad \begin{cases} \text{Optimize } F(x) \\ \text{Subject to } x \in X. \end{cases}$$

There are two types of criteria of solutions for the set-valued optimization problem, the vectorial criterion and set optimization criterion. The first criterion consists of looking for efficient points of the set $F(X) = \cup_{x \in X} F(x)$ and is called set-valued vector optimization problem. On the other hand, Kuroiwa et al. [3] started developing a new approach to set-valued optimization using the six types of set relations. The second criterion is based on comparison among values of F, that is, whole image $F(x)$ and seems to be more natural for set-valued optimization problem. In this paper, we treat the second type criterion and call set optimization problem.

The sublinear scalarizing functions for vectors [2,1] are generalized to set optimization problem by Shimizu et al. [4], Hamel–Löhne [5], Hernández–Rodríguez-Marín [6] and their references therein. Recently, Kuwano et al. [7] have introduced new unified approach on such scalarizations for sets.

In this paper, by mixing their idea, we investigate four types of nonlinear scalarizing functions for set valued maps and their relationships. We also give nonconvex separation type theorems for sets which are generalizations of [2,8,1]. Some applications of these functions are also given.

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The organization of this paper is as follows. In Section 2, we investigate several properties of nonlinear scalarizing functions for sets introduced by Kuwano et al. [7]. In Section 3, as applications, we first present Gordan's type alternative theorems for set-valued map which are generalizations of Nishizawa et al. [9]. We also give optimality conditions for set optimization problem which are generalizations of Gerth–Weidner [2]. We last generalize Takahashi's minimization theorem to set-valued map.

2. Mathematical preliminaries

In this section, let Y be a topological vector space and 0_Y be the origin of Y. For a set $A \subset Y$, $\operatorname{int} A$, $\operatorname{cor} A$ and $\operatorname{cl} A$ denote the topological interior, the algebraic interior and the topological closure of A, respectively.

2.1. Preliminaries of vector optimization

Let $C \subset Y$ be a closed convex cone, that is, $\operatorname{cl} C = C$, $C + C \subset C$ and $[0, \infty) \cdot C \subset C$. A cone C is called pointed if $C \cap (-C) = \{0_Y\}$ and solid if $\operatorname{int} C \neq \emptyset$.

It is well known that, given a pointed convex cone $C \subset Y$, we can induce a partial ordering $\leq_C$ in Y defined by $x \leq_C y$ when $y - x \in C$. We denote $x \leq_{\operatorname{int} C} y$ when $y - x \in \operatorname{int} C$, $x \nleq_C y$ when $y - x \notin C$ and $x \nleq_{\operatorname{int} C} y$ when $y - x \notin \operatorname{int} C$. This ordering is compatible with the vector structure of Y, that is, for every $x \in Y$ and $y \in Y$,

(i) $x \leq_C y$ implies that $x + z \leq_C y + z$ for all $z \in Y$
(ii) $x \leq_C y$ implies that $\alpha x \leq_C \alpha y$ for all $\alpha \geq 0$.

We say that a point $a \in A$ is a minimal [resp. weak minimal] point of A if there is no $\hat{a} \in A$ such that $\hat{a} \leq_C a$ [resp. $\hat{a} \leq_{\operatorname{int} C} a$]. When C is pointed, the above definition is equivalent to

$$A \cap (a - C) = \{a\} \quad [\text{resp. } A \cap (a - \operatorname{int} C) = \emptyset].$$

We denote by $\operatorname{Min}(A; C)$ [resp. $\operatorname{wMin}(A; \operatorname{int} C)$] the set of minimal [resp. weak minimal] points of A with respect to C [resp. $\operatorname{int} C$], respectively. We can easily see that

$$\operatorname{Min}(A; C) \subset \operatorname{wMin}(A; \operatorname{int} C) \subset A.$$

2.2. Preliminaries of set relations

We denote by $\mathcal{V}$ the family of nonempty subsets of Y. The sum of two sets $V_1, V_2 \in \mathcal{V}$ is defined by

$$V_1 + V_2 := \{v_1 + v_2 | v_1 \in V_1, v_2 \in V_2\}.$$

The product of $\alpha \in \mathbb{R}$ and $V \in \mathcal{V}$ is defined by

$$\alpha V := \{\alpha v | v \in V\}.$$

Then we can check that $\mathcal{V}$ is a vector space with $\{0_Y\}$ a zero element. We review some basic concepts of set-relation.

Definition 2.1 (*Set-Relations [3,10]*)**.** For nonempty sets $A, B \subset Y$ and a solid convex cone C in Y, we write

$$A \leq_C^l B \quad \text{by } B \subset A + C \qquad A \leq_{\operatorname{int} C}^l B \quad \text{by } B \subset A + \operatorname{int} C,$$
$$A \leq_C^u B \quad \text{by } A \subset B - C \qquad A \leq_{\operatorname{int} C}^u B \quad \text{by } A \subset B - \operatorname{int} C.$$

Remark 1. There are some differences between vector ordering and set ordering. In vector case, for $x, y \in Y$ and $C \subset Y$, we see that $y \in x + C$ and $x \in y - C$ are equivalent. On the other hand, in set case, for $A, B \in 2^Y$ and $C \subset Y$, we see that $B \subset A + C (A \leq_C^l B)$ and $A \subset B - C (A \leq_C^u B)$ are not equivalent. Next example shows the difference between $\leq_C^l$ and $\leq_C^u$.

Example 1. We set

$$Y = \mathbb{R}^2, \qquad C = \mathbb{R}_+^2 = \{(x, y) | x \geq 0, y \geq 0\}$$
$$A_1 = [0, 2] \times [0, 2] \qquad B_1 = [3, 5] \times [0, 1] \qquad A_2 = [0, 2] \times [1, 2] \qquad B_2 = [3, 5] \times [0, 2].$$

We can check that $A_1 \leq_C^l B_1$, $A_1 \nleq_C^u B_1$ and $A_2 \nleq_C^l B_2$, $A_2 \leq_C^u B_2$.

Therefore, $\leq_C^l$ and $\leq_C^u$ are not comparable.