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# Characterization of a monodromic singular point of a planar vector field

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ABSTRACT

The Newton diagram and the lowest-degree quasi-homogeneous terms of an analytic planar vector field allow us to determine whether an isolated singular point of the vector field is monodromic or has a characteristic trajectory.



## 1. Introduction

We are interested in the behavior of the trajectories in a neighborhood of a singular point of the planar analytic differential system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{1}$$

and, in particular, in determining when a singular point (we can assume the origin to be the singular point) is surrounded by orbits of the system (monodromic singular point).

Each trajectory by lying on a vicinity of a monodromic singular point is either a spiral or an oval. Moreover, from the finiteness theorem for the number of limit cycles, a monodromic point of an analytic planar vector field can be only either a focus or a center, see Il'yashenko [1]. So, the monodromy problem is a prior step to solving the center problem of a vector field which is one of the classical open problems in the qualitative theory of planar differential systems.

If the differential matrix $D\mathbf{F}(\mathbf{0})$ is not identically null, the monodromy problem is completely solved. The problem when the eigenvalues of the matrix are conjugated complex, was solved by Poincaré [2] and when the matrix is nilpotent, by Andreev [3]. Finally, if $D\mathbf{F}(\mathbf{0})$ is identically null (in such a case, $O$ is a degenerate singular point), the monodromy problem can be solved by using the blow-up technique (developed by Dumortier [4]) which consists of performing a series of changes to desingularize the point. However, its application for determining the monodromy of a singular point of a family of vector fields with parameters becomes rather complicated. Some works that use this technique in order to study the monodromy are [5–8]. All of them are only partial results.

In order to show our results, we need to recall the following concepts that we will use throughout the paper: the quasi-homogeneous vector fields (in particular, the conservative–dissipative splitting of a quasi-homogeneous vector field), the Newton diagram of a vector field and the generalized polar coordinates, introduced by Liapunov [9].

*Conservative–dissipative splitting*

Let $\mathbf{t} = (t_1, t_2)$ be non-null with $t_1$ and $t_2$ non-negative integer numbers without common factors. A function $f$ of two variables is quasi-homogeneous of type $\mathbf{t}$ and degree $k$ if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x, y)$. The vector space of quasi-homogeneous

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polynomials of type $\mathbf{t}$ and degree $k$ will be denoted by $\mathscr{P}_k^{\mathbf{t}}$. A vector field $\mathbf{F} = (F_1, F_2)^T$ is quasi-homogeneous of type $\mathbf{t}$ and degree $k$ if $F_1 \in \mathscr{P}_{k+t_1}^{\mathbf{t}}$ and $F_2 \in \mathscr{P}_{k+t_2}^{\mathbf{t}}$. We will denote $\mathscr{Q}_k^{\mathbf{t}}$ the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree $k$.

The quasi-homogeneous vector monomials can be determined by drawing the lattice $\mathbb{Z}_+^2$, and assigning each point $(m, n)$ to the quasi-homogeneous vector fields $(x^m y^{n-1}, 0)^T$ and $(0, x^{m-1}y^n)^T$. The points with integer coordinates aligned in the straight lines perpendicular to $\mathbf{t}$, $(m-1)t_1 + (n-1)t_2 = k$, determine the quasi-homogeneous vector monomials with the same degree $k$.

Any vector field can be expanded into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. Thus, the system (1) can be written in the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots = \sum_{j=0}^{\infty} \mathbf{F}_{r+j}(\mathbf{x}),$$

for some $r \in \mathbb{Z}$, where $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathscr{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r \not\equiv \mathbf{0}$. These expansions are usually considered in the analysis of the topological determination of the singularity by means of the blow-up technique (see [10,11,4]). This concept also has been used by Algaba et al. [12] as an application of the Normal Form Theory, and for the study of the integrability and of the center problem of systems with a singular point degenerated, i.e. systems whose matrix of the linear part evaluated in the singular point is identically null, see [13,14].

Next, we cite the splitting of a quasi-homogeneous vector field as a sum of two quasi-homogeneous vector fields, a conservative one (having zero-divergence) and a dissipative one (in the sense of the non-conservative part that fully captures the divergence of the vector field) that will be useful in what follows and will play a main role in our analysis. Throughout this paper, the Hamiltonian system associated to the $\mathcal{C}^1$ function $f$ is denoted by $\mathbf{X}_f$, i.e. $\mathbf{X}_f = (-\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x})^T$. Algaba et al. [14] proved that any quasi-homogeneous vector field $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathscr{Q}_j^{\mathbf{t}}$ can be expressed as

$$\mathbf{F}_j = \mathbf{X}_{h_{j+|\mathbf{t}|}} + \mu_j \mathbf{D}_0, \tag{2}$$

where $\mathbf{D}_0(x, y) := (t_1 x, t_2 y)^T$ (a dissipative quasi-homogeneous vector field of type $\mathbf{t}$ and degree 0), $\mu_j := \frac{1}{j+|\mathbf{t}|}\operatorname{div}(\mathbf{F}_j) \in \mathscr{P}_j^{\mathbf{t}}$ (the divergence of $\mathbf{F}_j$), $h_{j+|\mathbf{t}|} := \frac{1}{j+|\mathbf{t}|}(t_1 x Q_{j+t_2} - t_2 y P_{j+t_1}) \in \mathscr{P}_{j+|\mathbf{t}|}^{\mathbf{t}}$ (the wedge product of $\mathbf{D}_0$ and $\mathbf{F}_j$) and $|\mathbf{t}| = t_1 + t_2$.

We note that any non-vanishing quasi-homogeneous polynomial of type $\mathbf{t} = (t_1, t_2)$ with $t_1$ and $t_2$ non-null, in particular $h_{j+|\mathbf{t}|}$, can be expressed as $p(x, y) = x^{k_1} y^{k_2} p_0(x^{t_2}, y^{t_1})$ with $0 \le k_1 < t_2$, $0 \le k_2 < t_1$ being $p_0$ a homogeneous polynomial. So, by abusing the notation, we can write any quasi-homogeneous polynomial of type $\mathbf{t}$ in a compact form $p(x, y) = c \prod_{j=0}^{m} f_j^{m_j} \prod_{j=0}^{n} g_j^{n_j}$, where

$$f_j(x, y) = x,\ y \quad \text{or} \quad y^{t_1} - \lambda_j x^{t_2}, \quad j = 0, \ldots, m$$

and

$$g_j(x, y) = (y^{t_1} - a_j x^{t_2})^2 + b_j^2 x^{2t_2}, \quad j = 0, \ldots, n$$

with $c$, $\lambda_j$, $a_j$ and $b_j$ real numbers and $\lambda_j$, $b_j$ non-zero, for all $j$.
If $h_{r+|\mathbf{t}|} \in \mathscr{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ and $\mu_r \in \mathscr{P}_r^{\mathbf{t}}$ are the polynomials associated to the lowest-degree quasi-homogeneous term of type $\mathbf{t}$ of $\mathbf{F}$, we will say that a polynomial of the form $x$, $y$ or $y^{t_1} - \lambda x^{t_2}$, $\lambda \neq 0$, is a *strong factor of* $\mathbf{F}$ *associated to the type* $\mathbf{t}$, or simply a *strong factor of* $h_{r+|\mathbf{t}|}$, if it satisfies one of the following properties:

(i) it is a factor of $h_{r+|\mathbf{t}|}$ of odd multiplicity order,
(ii) it is a factor of $h_{r+|\mathbf{t}|}$ of even multiplicity order $(2m)$ and, either it is not a factor of $\mu_r$ with $\mu_r \not\equiv 0$ or is a factor of $\mu_r$ with even multiplicity order $(2n)$ with $0 < n < m$.

*Newton diagram*

We will write the components of the vector field $\mathbf{F}$ in the form $P(x, y) = \sum a_{ij} x^i y^{j-1}$ and $Q(x, y) = \sum b_{ij} x^{i-1} y^j$.
The *support* of (1) and also of $\mathbf{F}$, denoted by *supp*$(\mathbf{F})$, is the set of pairs $(i, j)$ with $(a_{ij}, b_{ij}) \neq (0, 0)$. The vector $(a_{ij}, b_{ij})$ is called the *vector coefficient* of $(i, j)$ in the support. Consider the set

$$\bigcup_{(i,j) \in supp(\mathbf{F})} \left((i, j) + \mathbb{R}_+^2\right),$$

where $\mathbb{R}_+^2$ is the positive quadrant and the union is taken over all points $(i, j)$ in the support. The boundary of the convex hull of this set is made up of two open rays and a polygon, which can be just one point. The polygon together with the rays that do not lie on a coordinate axis, if they existed, is called the *Newton diagram* of the vector field $\mathbf{F}$. The component parts of the Newton diagram are called *edges* and their endpoints are the *vertices* of the Newton diagram.