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Existence of quasi-periodic solutions and Littlewood's boundedness problem of sub-linear impact oscillators

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ABSTRACT

In this paper, it is shown by a series of transformations that how Moser's invariant curve theorem can be used to analyze the dynamical behavior of sub-linear Duffing-type equations with impact. We prove that all solutions are bounded, and that there are infinitely many periodic and quasi-periodic solutions in this impact case.

1. Introduction and main results

Impact oscillators of the form

$$\begin{cases} \ddot{x} + V_x'(x,t) = 0, & \text{for } x(t) > 0; \\ x(t) \geq 0; \\ \dot{x}(t_0^+) = -\dot{x}(t_0^-), & \text{if } x(t_0) = 0, \end{cases} \tag{1}$$

where $\dot{x}$ denotes dx/dt, serve as models of dynamical systems with discontinuity; cf. [1]. From the viewpoint of mechanics, these types of equations model the motion of a particle attached to a nonlinear spring and bouncing elastically against the fixed barrier ($x = 0$); cf. Fig. 1.

If $V_x'(0,t) \equiv 0$, Eq. (1) could always be simplified to a second order equation without impact by a centro-symmetric vector field. But many typical models could not be discussed as above. For example, the forced impact oscillators ($V_x'(0,t) = g(x) - p(t)$) do not satisfy that $V_x'(0,t) \equiv 0$.

Systems of the form (1) are special cases of vibro-impact systems; see e.g., [2]. They are also related to the Fermi accelerator, see e.g., [3], dual billiards, see e.g., [4], and certain models used in celestial mechanics, see e.g., [5]. The nonsmoothness caused by the impact limits the applications of many useful mathematical tools. However, there are still some interesting papers on the periodic, quasi-periodic and other regular motions for the impact oscillators; see [3,4,6–14] and the references therein.

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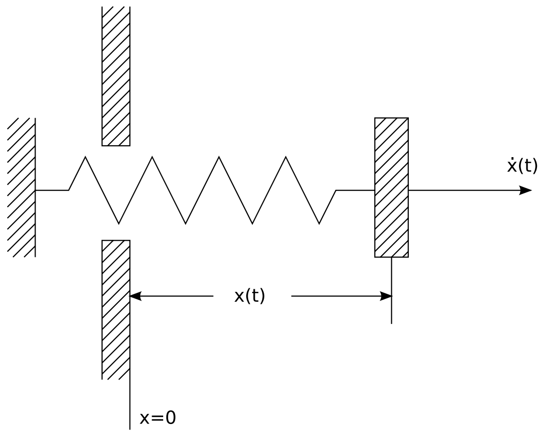

Fig. 1. Impact oscillators of Eq. (1).

Different from the existence of periodic solutions, people must overcome the nonsmoothness of x at $x = 0$, to apply variant versions of Moser's invariant curve theorem to obtain the existence and multiplicity of quasi-periodic solutions followed by the existence of invariant tori.

Specially, in [14], Zharnitsky construct a Hamiltonian, which belongs to C^0 with respect to the angle variable, defined in the whole phase plane by centro-symmetric vector field. Exchanging the roles of time and angle variables, the Poincaré mapping of the new Hamiltonian vector field which is analytic in the new action and angle variables is closed to an integrable one and the perturbation is of $O(\epsilon)$ by rescaling the system. Then Moser's invariant curve theorem [15] guarantees the existence of arbitrarily large invariant tori.

Ortega [8] and Qian and Sun [10] consider the linear and asymptotically linear impact oscillators, e.g., $V'_x(x,t) = a^2x + \phi(x) - p(t)$, where $\phi(x)$ satisfies some assumptions. The nonsmoothness is overcome by the successor map. Then the existence of invariant tori are proved by a variant version of Moser's small twist theorem given by Ortega [16,17].

In this paper, we will discuss the sub-linear impact oscillators. Consider a typical sub-linear case $V'_x(x,t) = x|x|^{\alpha-1} - p(t),\ 0 < \alpha < 1$, i.e.,

$$\begin{cases} \ddot{x} + x|x|^{\alpha-1} = p(t), & \text{for } x(t) > 0; \\ x(t) \geq 0; \\ \dot{x}(t_0^+) = -\dot{x}(t_0^-), & \text{if } x(t_0) = 0, \end{cases} \tag{2}$$

where $p(t)$ is a C^∞ periodic function with period 1.

Different from the case in [14] (the perturbation seems to be analytic), the nonsmoothness on x at $x = 0$ and the smoothness condition on $p(t)$ will cause some extra difficulties in our proof.

More precisely, we have

Theorem 1.1. *Suppose $p(t)$ is a C^∞ 1-periodic function and $0 < \alpha < 1$. Then every solution of Eq. (2) is bounded, i.e., for every $(t_0, x_0, \dot{x}_0)$,*

$$\sup_{t\in\mathbb{R}}(|x(t;t_0,x_0,\dot{x}_0)| + |\dot{x}(t;t_0,x_0,\dot{x}_0)|) < \infty.$$

Moreover, there exist infinitely many quasi-periodic solutions with large amplitude of the form

$$x(t) = f(\lambda t, t),$$

where f is defined on a 2-torus and λ^{-1} is a large Diophantine irrational number.

Remark 1.1. The main difficulty compared with [14] is that the perturbation g in new Hamiltonian (see (13)), is of C^0 class with respect to τ caused by impact at $x = 0$ and not analytic with the variable θ, so g is not small enough to use Moser's theorem. Following Kupper and You [18], the term g must be at least of C^2 class with respect to τ if we want to do some transformations to make the perturbation small enough.

Fortunately, we can do a C^2 approximation of g (cf. Section 3), and leave the C^0 term small enough (this trick is first used in [19] to relax further the smooth requirement of periodic coefficients $p_j(t)$'s). Then we can use Moser's theorem [15] to obtain the invariant tori. For similar results about sub-linear oscillators without impact, see [18,20–22], etc.

Remark 1.2. The existence of infinitely many periodic solutions is a direct conclusion of Theorem 1.1 and Poincaré–Birkhoff twist fixed-point theorem. Since there are many results in this direction, we omit it; cf. Dieckerhoff and Zehnder [23], etc.