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Nonexistence of solutions for singular elliptic equations with a quadratic gradient term ☆

Wenshu Zhou [a], Xiaodan Wei [b,*], Xulong Qin [c]

[a] Department of Mathematics, Dalian Nationalities University, Dalian 116600, China
[b] College of Computer Science, Dalian Nationalities University, Dalian 116600, China
[c] Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, China

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ABSTRACT

This paper concerns the nonexistence of solutions for singular elliptic equations with a quadratic gradient term. The main results complement and partly extend some works by Arcoya et al. (2009) [1]. As a by-product of the main results, we fill in a gap in one of their works.

1. Introduction

In the paper, we consider the elliptic problem:

$$\begin{cases} -\Delta u + g(x,u)|\nabla u|^2 = f(x), & u > 0, \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded set of $\mathbb{R}^N (N \geq 1)$, $f(x)$ is measurable in Ω, and $g : \Omega \times (0, +\infty) \to \mathbb{R}$ is a Carathéodory function such that $g(x,s) \geq 0$ for a.e. $x \in \Omega$ and all $s > 0$, moreover, $g(\cdot, s)$ may be singular at $s = 0$, whose simplest model is

$$\begin{cases} -\Delta u + \dfrac{|\nabla u|^2}{u^\gamma} = f(x), & u > 0, \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\gamma > 0$.

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* Corresponding author. Tel.: +86 411 87189428; fax: +86 411 87189428.
E-mail address: weixiaodancat@126.com (X. Wei).

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Recently, such problems have been extensively studied in the literature, for example [1–8] and references therein, where some results on existence, nonexistence and uniqueness of solutions for problem (1.1) were obtained. It is worth pointing out that problem (1.1) may have multiple solutions in general (see [8]). Some related studies can be referred to [10,9,11–16] and the references cited in the paper mentioned above.

Motivated by the paper [1], where the authors proved some results on the existence and nonexistence of finite energy solutions for problem (1.1), in the present paper, we will mainly discuss the issue of the nonexistence of finite energy solutions for problem (1.1). Following [1], we say that a function $u \in H_0^1(\Omega)$ is a finite energy solution for problem (1.1) if it satisfies $u > 0$ a.e. in Ω and $g(x,u)|\nabla u|^2 \in L^1(\Omega)$, and solves the equation in the sense of distributions. Our results complement and partly extend the corresponding works of [1]. As far as problem (1.2) is concerned, moreover, we fill in the gap in the result [1, Corollary 4.5] where the case $\gamma = 2$ is not settled completely for the nonexistence or existence. Indeed, we prove that, without any additional requirement of f, $\gamma = 2$ falls in the interval of nonexistence (see Corollary 1.1).

Let us recall some main results of [1]. In [1, Theorem 1.1], the authors stated that there exists a finite energy solution for problem (1.1) provided that $f \in L^{\frac{2N}{2+N}}(\Omega)$ $(N \geq 3)$ and

$$\operatorname{essinf}\{f(x) : x \in \omega\} > 0, \quad \forall \omega \Subset \Omega, \tag{1.3}$$

and that g satisfies

$$0 \leq g(x,s) \leq h(s) \quad \text{for a.e. } x \in \Omega \text{ and for all } s > 0, \tag{1.4}$$

where $h : (0, +\infty) \to [0, +\infty)$ is a nonnegative continuous function such that

$$\begin{cases} \lim_{s\to 0^+} \int_s^1 \sqrt{h(t)}dt < +\infty, \\ h(s) \text{ is nonincreasing near a neighborhood of zero.} \end{cases} \tag{1.5}$$

Furthermore, they handled the case where $f \in L^1(\Omega)$ and proved an existence result.

For studying the nonexistence, they assume that g satisfies

$$0 \leq h(s) \leq g(x,s) \quad \text{for a.e. } x \in \Omega \text{ and for all } s > 0, \tag{1.6}$$

where $h : (0, +\infty) \to [0, +\infty)$ is a nonnegative continuous function such that

$$\lim_{s\to 0^+} h(s) = +\infty, \qquad \lim_{s\to 0^+} \int_s^1 \sqrt{h(t)}dt = +\infty, \tag{1.7}$$

and

$$\lim_{s\to 0^+} \sqrt{h(s)}e^{\int_1^s \sqrt{h(t)}dt} =: h_0 \geq 0. \tag{1.8}$$

They proved the following.

Theorem A *([1, Theorem 1.4]). Let $0 \leq f \in L^q(\Omega)$ with $q > \frac{N}{2} (N \geq 3)$ and $f \not\equiv 0$. Suppose that (1.6)–(1.8) hold. If $\lambda_1(f) > 1$, then problem (1.1) has no finite energy solution. Here $\lambda_1(f)$ denotes the first positive eigenvalue of the operator $-\Delta$ with homogeneous Dirichlet boundary conditions.*

Theorem B *([1, Corollary 4.5]). Let $0 \leq f \in L^q(\Omega)$ with $q > \frac{N}{2} (N \geq 3)$ and $f \not\equiv 0$. Suppose that there exist some constants $s_0 > 0$, $\Lambda > 0$, $\gamma \geq 2$ such that*

$$g(x,s) \geq \frac{\Lambda}{s^\gamma} \quad \textit{for a.e. } x \in \Omega \textit{ and for all } s \in (0, s_0]. \tag{1.9}$$

If either $\gamma > 2$ or $\gamma = 2$ and $\lambda_1(f) > \frac{1}{\Lambda}$, then problem (1.1) has no finite energy solution.

As an immediate consequence of [1, Theorem 1.1] and Theorem B, they showed that problem (1.2) has a finite energy solution for every $f \in L^q(\Omega)$ with $q > \frac{N}{2} (N \geq 3)$ satisfying (1.3) if and only if $\gamma < 2$ (see [1, Theorem 1.5]). It is clear that the above claim holds only in some sense since the nonexistence does not cover the case where $\gamma = 2$ and $\lambda_1(f)$ is sufficiently small.

Note that in the above conclusions, there is a restriction of regularity of f. In the present paper, without any additional requirement on f, we establish the following nonexistence result by a simple proof.

Theorem 1.1. *Let h satisfy (1.7). We have the following.*

(1) *If there exist some positive constants ε_0, C_1, C_2 such that*

$$h(s) \leq C_1 h(t) + C_2, \quad \forall 0 < t < s < \varepsilon_0, \tag{1.10}$$