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Bogolyubov-type theorem with constraints induced by a control system with hysteresis effect[☆]

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ABSTRACT

We consider the problem of minimization of an integral functional with nonconvex with respect to the control integrand. We minimize our functional over the solution set of a control system described by two ordinary differential equations subject to a control constraint given by a multivalued mapping with closed nonconvex values. The coefficients of the equations and the constraint depend on the phase variables. One of the equations contains the subdifferential of the indicator function of a closed convex set depending on the unknown phase variable. The equation containing the subdifferential describes an input–output relation of hysteresis type.

Along with the original problem, we also consider the problem of minimizing the integral functional with the convexified with respect to the control integrand over the solution set of the same system with the convexified control constraint.

Under sufficiently general assumptions, we prove that this relaxed problem has an optimal solution, which is the limit of a minimizing sequence of the original problem. The convergence takes place simultaneously with respect to the trajectory, the control and the functional and is uniform in appropriate topologies.

1. Introduction

In the article [1] dating back to 1930, Bogolyubov proved a theorem on relaxation for a class of problems of the classical calculus of variations. Since then this theorem has been extended in several directions by many authors including Young [2], MacShane [3], Ioffe and Tikhomirov [4], Ekeland and Temam [5]. Among more recent generalizations are the works by De Blazi, Pianigiani and Tolstonogov [6], Tolstonogov [7–9]. The main objective of the present paper is to prove an analogue of Bogolyubov's theorem for the following problem.

Let T be the interval $[0, 1]$ of the real line $\mathbb{R}$ with the Lebesgue measure μ and the σ-algebra Σ of μ-measurable subsets of T, $\mathbb{R}^+ = [0, +\infty)$, $\overline{\mathbb{R}} = (-\infty, +\infty]$. For a function $g : T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ consider the problem

$$\int_T g(t, v(t), w(t), u(t))\, dt \to \inf \tag{P}$$

on the solution set of a nonlinear control system described by two ordinary differential equations of the following form

$$a_1(v(t), w(t))v'(t) + a_2(v(t), w(t))w'(t) = h_1(v(t), w(t))u^1(t) + c_1(v(t), w(t)), \tag{1.1}$$

$$b_1(v(t), w(t))v'(t) + b_2(v(t), w(t))w'(t) + \partial I_{v(t)}(w(t)) \ni h_2(v(t), w(t))u^2(t) + c_2(v(t), w(t)), \tag{1.2}$$

$$v(0) = v_0, \qquad w(0) = w_0,$$

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$t \in T$, subject to the mixed control constraint

$$u(t) = \big(u^1(t), u^2(t)\big) \in U(t, v(t), w(t)) \quad \text{a.e. on } T. \tag{1.3}$$

Here $a_i(\cdot,\cdot)$, $b_i(\cdot,\cdot)$, $c_i(\cdot,\cdot)$, $h_i(\cdot,\cdot)$, $i = 1, 2$, are scalar functions; for each $v \in \mathbb{R}$, $\partial I_v(\cdot)$ is the subdifferential of the indicator function $I_v(\cdot)$ of the interval $[f_*(v), f^*(v)] \subset \mathbb{R}$ with $f_*(\cdot)$ and $f^*(\cdot)$ being two nondecreasing functions such that $f_* \le f^*$ on $\mathbb{R}$; U is a multivalued mapping with compact values; v_0 and w_0 are given numbers such that

$$f_*(v_0) \le w_0 \le f^*(v_0). \tag{1.4}$$

Note that the inclusion (1.2) describes plenty of physically relevant input–output relations $v \to w$ of hysteresis type. For instance, when $b_1 \equiv 0, b_2 \equiv 1, h_1 \equiv 0, c_2 \equiv 0$ and $b_1 \equiv -1, b_2 \equiv 1, h_1 \equiv 0, c_2 \equiv 0$, the inclusion (1.2) models the generalized play operator and the generalized stop operator, respectively (cf. [10–12]). These operators are typical examples of hysteresis input–output relations, and are used for the analysis of many nonlinear irreversible phenomena in nature such as solid–liquid phase transition with undercooling/superheating effect [13–15], real-time control problems [16], shape memory alloy problems [17], filtration problems [18], problems from population dynamics [19], etc.

Let $g_U : T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \overline{\mathbb{R}}$ be defined as

$$g_U(t, v, w, u) = \begin{cases} g(t, v, w, u), & u \in U(t, v, w), \\ +\infty, & u \notin U(t, v, w). \end{cases} \tag{1.5}$$

and $g_U^{**}(t, v, w, u)$ be the bipolar (the second conjugate) of the function $u \mapsto g_U(t, v, w, u)$ [5].

Along with (P) we consider the following relaxed problem

$$\int_T g_U^{**}(t, v(t), w(t), u(t))\, dt \to \min \tag{RP}$$

on the solution set of the control system (1.1), (1.2) with the convexified control constraint

$$u(t) \in \text{co}U(t, v(t), w(t)) \quad \text{a.e. on } T, \tag{1.6}$$

where $\text{co}U(t, v(t), w(t))$ is the convex hull of the set $U(t, v(t), w(t))$, $t \in T$.

In this paper we explore interrelations between the solutions of the problems (P) and (RP). Under sufficiently general assumptions, we prove that for any solution (v_*, w_*, u_*) of the control system (1.1), (1.2) with the constraint (1.6), there exists a sequence of solutions (v_n, w_n, u_n), $n \ge 1$, of the control system (1.1), (1.2) with the constraint (1.3), such that in appropriate topologies we have the following convergences

$$(v_n, w_n, u_n) \to (v_*, w_*, u_*), \tag{1.7}$$

$$\int_T g(t, v_n(t), w_n(t), u_n(t))\, dt \to \int_T g_U^{**}(t, v_*(t), w_*(t), u_*(t))\, dt, \tag{1.8}$$

In addition, we show that Problem (RP) has a solution, and for any solution (v_*, w_*, u_*) of (RP), there exists a minimizing sequence (v_n, w_n, u_n), $n \ge 1$, of Problem (P) for which (1.7), (1.8) hold. This property is usually called the relaxation [5]. The relations (1.7), (1.8) are an analogue of Bogolyubov's theorem with constraints given by the sets of solutions of the control systems (1.1)–(1.3) and (1.1), (1.2), (1.6).

In our proofs we follow the ideas of the article [7]. Taking advantage of the fact that the problem treated in the present paper is considered on a finite dimensional space we are able to weaken some hypotheses on the integrand function g as compared to [7], in particular we do not assume that g is Lipschitz continuous.

2. Preliminary notions and main assumptions

Let $\|\cdot\|$ and $d(\cdot,\cdot)$ denote the norm and the distance, respectively, on the Euclidean space $\mathbb{R}^2$. The Hausdorff metric on the space of compact subsets in $\mathbb{R}^2$ we denote by $D(\cdot,\cdot)$. For a Banach space X the notation ω-X means that the space X is equipped with the weak topology. The same notation is used for subsets of the space X with the topology induced by that of the space ω-X.

Let Y be a metric space. A multivalued mapping $F : Y \to \mathbb{R}^2$ with compact values is called continuous if it is continuous with respect to the Hausdorff metric $D(\cdot,\cdot)$.

A multivalued mapping $F : X \to X$ is called Vietoris lower semi-continuous at a point $x_0 \in X$ if for any open set $V \subset X$, $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood $V(x_0)$ of the point x_0, such that $F(x) \cap V \neq \emptyset$ for every $x \in V(x_0)$. It is known that a mapping $F : X \to X$ is Vietoris lower semi-continuous if and only if the mapping $\overline{F} : X \to X$, $\overline{F}(x) = \overline{F(x)}$, is Vietoris lower semi-continuous, where the bar denotes the closure in the strong topology of X.

A multivalued mapping $F : T \to X$ is called measurable [20] if $F^-(V) = \{t \in T : F(t) \cap V \neq \emptyset\} \in \Sigma$, for any open set $V \subset X$. If $F : T \times X \to X$, then the $\Sigma \otimes \mathcal{B}(X)$-*measurability* means that $F^-(V) \in \Sigma \otimes \mathcal{B}(X)$ for any open $V \subset X$, where $\Sigma \otimes \mathcal{B}(X)$ is the σ-algebra of subsets of $T \times X$ generated by the sets $A \times B$, $A \in \Sigma$, $B \in \mathcal{B}(X)$, and $\mathcal{B}(X)$ is the σ-algebra of Borel sets of X.