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Multiple positive solutions of non-local initial value problems for first order differential systems

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ABSTRACT

The paper gives a new and natural method for the existence of multiple positive solutions for first order differential systems with non-local initial value conditions involving linear functionals. The case of higher order differential equations is also considered. The results are accompanied by numerical examples confirming the theory and proving for practice the importance of the bounds in solution localization.

1. Introduction

In this paper we deal with the existence, localization and multiplicity of positive solutions to non-local problems for first order differential systems, namely

$$\begin{cases} u' = f(t, u), & t \in [0, 1] \\ u(0) = \alpha[u], \end{cases} \tag{1.1}$$

where $f : [0, 1] \times \mathbf{R}^n_+ \to \mathbf{R}^n$ is continuous and $\alpha : C([0, 1], \mathbf{R}^n) \to \mathbf{R}^n$ is linear continuous. We shall seek solutions $u \in C^1([0, 1], \mathbf{R}^n) \cap C\left([0, 1], \mathbf{R}^n_+\right)$. In particular, we shall obtain results for non-local initial value problems related to n-order differential equations, namely for

$$\begin{cases} x^{(n)} = g\left(t, x, x', \dots, x^{(n-1)}\right), & t \in [0, 1] \\ x(0) = \alpha_1\left[x, x', \dots, x^{(n-1)}\right], \\ x'(0) = \alpha_2\left[x, x', \dots, x^{(n-1)}\right], \\ \dots \\ x^{(n-1)}(0) = \alpha_n\left[x, x', \dots, x^{(n-1)}\right]. \end{cases} \tag{1.2}$$

Such kind of problems arise from mathematical modeling of real processes, for instance heat, fluid, chemical or biological flow, where non-local conditions can be interpreted as feedback controls.

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Non-local problems have been extensively studied mainly with multi-point boundary conditions (see, e.g., [1–4] and references therein for first order equations, [4–9] for second-order equations, and [10,11] for higher order equations). Problems with boundary conditions given by linear continuous functionals, equivalently, by Stieltjes integrals were studied in [7,11–14].

One of the most used approaches to studying positive solutions of a boundary value problem is to rewrite it as an integral equation and to take advantage from the good properties of the corresponding Green function. More exactly, upper and lower estimations of Green's function are enough to define a smaller cone of positive functions where Krasnoselskii type compression–expansion theorems apply. Thus the problem reduces to finding the corresponding Green's function, which in many cases is a laborious and tiring task, and in establishing its bilateral, upper and lower bounds. In the present paper, no Green functions appear and we just manage the localization of solutions by means of their extrema.

We conclude this introductory section by some notations. For two vectors $r, R \in \mathbf{R}^n$, $r = (r_1, r_2, \ldots, r_n)$, $R = (R_1, R_2, \ldots, R_n)$, we let $r \leq R$ (and $r < R$) if $r_i \leq R_i$ (respectively $r_i < R_i$) for $i = 1, 2, \ldots, n$. Also we shall denote by $\alpha[1]$ the matrix whose columns are

$$\alpha[(1, 0, \ldots, 0)], \alpha[(0, 1, \ldots, 0)], \ldots, \alpha[(0, 0, \ldots, 1)],$$

and by $\alpha[t]$, the matrix of columns

$$\alpha[(t, 0, \ldots, 0)], \alpha[(0, t, \ldots, 0)], \ldots, \alpha[(0, 0, \ldots, t)].$$

Here, for example, $(1, 0, \ldots, 0)$ stands as a constant vector-valued function and $(t, 0, \ldots, 0)$ for the vector-valued function of scalar components $t, 0, \ldots, 0$. Thus, for any constant vector-valued function $c = (c_1, c_2, \ldots, c_n)$, we decompose

$$c = c_1 (1, 0, \ldots, 0) + c_2 (0, 1, \ldots, 0) + \cdots + c_n (0, 0, \ldots, 1)$$

and so we obtain that

$$\alpha[c] = \alpha[1]\, c,$$

where c is looked as column vector. Let us make the convention that all vectors in $\mathbf{R}^n$ are identified to column matrices.

Finally, for a scalar function $v \in C[0, 1]$, we shall denote by $|v|_\infty$ its supremum norm, and for a vector-valued function $u \in C([0, 1], \mathbf{R}^n)$, of scalar components $u_1, u_2, \ldots, u_n$, we shall denote by $|u|_\infty$ the column vector of elements $|u_1|_\infty, |u_2|_\infty, \ldots, |u_n|_\infty$.

2. The non-resonance positone case

First we consider problem (1.1) under the *non-resonance* condition:

$$\text{matrix } I - \alpha[1] \text{ is non-singular.} \tag{2.1}$$

Under assumption (2.1), problem (1.1) is equivalent to the integral system

$$u(t) = (I - \alpha[1])^{-1} \alpha\left[\int_0^t f(s, u(s))\, ds\right] + \int_0^t f(s, u(s))\, ds, \quad t \in [0, 1],$$

which can be seen as a fixed point problem in $C\left([0, 1], \mathbf{R}^n_+\right)$, for the operator $N : C\left([0, 1], \mathbf{R}^n_+\right) \to C([0, 1], \mathbf{R}^n)$, given by

$$(Nu)(t) = (I - \alpha[1])^{-1} \alpha\left[\int_0^t f(s, u(s))\, ds\right] + \int_0^t f(s, u(s))\, ds. \tag{2.2}$$

In what follows we shall denote by $N_i u$, $i = 1, 2, \ldots, n$ the scalar components of Nu.

We also assume in this section the *positone* case characterized by the following three conditions:

$$f \text{ is non-negative, i.e., } f\left([0, 1] \times \mathbf{R}^n_+\right) \subset \mathbf{R}^n_+;$$

$$\alpha[u] \in \mathbf{R}^n_+ \quad \text{for all } u \in C^1([0, 1], \mathbf{R}^n) \text{ with } u' \in C\left([0, 1], \mathbf{R}^n_+\right) \text{ and } u(0) = 0; \tag{2.3}$$

$$\begin{aligned}&\text{matrix } I - \alpha[1] \text{ is inverse positive, i.e.,}\\ &\text{the elements of its inverse are non-negative.}\end{aligned} \tag{2.4}$$

Then, in view of (2.2) one has that N maps the positive cone $C\left([0, 1], \mathbf{R}^n_+\right)$ into itself. Also, it is a simple remark that the positivity of f implies that for any $u \in C\left([0, 1], \mathbf{R}^n_+\right)$, Nu is increasing and consequently, $|Nu|_\infty = (Nu)(1)$ and

$$(Nu)(t) \geq (Nu)(0), \quad t \in [0, 1]. \tag{2.5}$$

This suggests the definition of a sub-cone of $C\left([0, 1], \mathbf{R}^n_+\right)$, namely

$$K := \{u \in C\left([0, 1], \mathbf{R}^n_+\right) : u(t) \geq u(0) \text{ for all } t \in [0, 1]\}.$$

Clearly, (2.5) shows that $N(K) \subset K$.

For any vectors $r \in \mathbf{R}^n_+$ and $R \in (0, \infty)^n$, consider the set

$$K_{rR} := \{u \in K : r \leq u(0) \text{ and } |u|_\infty \leq R\}.$$