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A multiplicity result for periodic solutions of second order differential equations with a singularity

Alberto Boscaggin [a], Alessandro Fonda [b,*], Maurizio Garrione [a]

[a] *SISSA - International School for Advanced Studies, Via Bonomea 265, I-34136 Trieste, Italy*

[b] *Dipartimento di Matematica e Informatica, Università di Trieste, P.le Europa 1, I-34127 Trieste, Italy*

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ABSTRACT

By the use of the Poincaré–Birkhoff fixed point theorem, we prove a multiplicity result for periodic solutions of a second order differential equation, where the nonlinearity exhibits a singularity of repulsive type at the origin and has linear growth at infinity. Our main theorem is related to previous results by Rebelo (1996, 1997) [4,5] and Rebelo and Zanolin (1996) [6,7], in connection with a problem raised by del Pino et al. (1992) [1].

1. Introduction

In [1], del Pino, Manásevich and Montero considered an equation like

$$x'' - \frac{1}{x^{\nu}} + \beta x = p(t), \tag{1.1}$$

where $p : \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic, $\nu \geq 1$, and $\beta > 0$. They proved that, if

$$\beta \neq \left(\frac{k\pi}{T}\right)^2, \quad \text{for every } k \in \mathbb{N}, \tag{1.2}$$

then there exists at least one positive T-periodic solution to (1.1). Their result followed the path opened by Lazer and Solimini in [2], where the case $\beta = 0$ was analyzed. It was shown there that, in this case, a necessary and sufficient condition for the existence of a positive T-periodic solution is that $\int_0^T p(t)\, dt < 0$.

In general, condition (1.2) is not eliminable. Indeed, if $\beta = \left(\frac{k\pi}{T}\right)^2$ for some positive integer k, some kind of resonance can occur: as shown in [3, Theorem 3], taking $p(t) = \epsilon \sin(\frac{2\pi k}{T} t)$, with $|\epsilon|$ sufficiently small, no T-periodic solutions to (1.1) can exist.

Quoting the last sentence in [1],

> "*…the solution we are predicting in our "Fredholm alternative" for* (1.1) *is not necessarily unique, so the multiplicity problem for this simple equation is raised as an open question*".

* Corresponding author.
E-mail addresses: boscaggi@sissa.it (A. Boscaggin), a.fonda@units.it (A. Fonda), garrione@sissa.it (M. Garrione).

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In [4–7], Rebelo and Zanolin analyzed the multiplicity problem assuming the forcing term to be of the form $p(t) = s+e(t)$, being s a real parameter. By the use of the Poincaré–Birkhoff fixed point theorem, they proved that, for $|s|$ large enough, Eq. (1.1) may have a large number of T-periodic solutions. Their results apply to the wider class of T-periodic problems of the type

$$\begin{cases} x'' + h(x) = s + e(t) \\ x(0) = x(T), \qquad x'(0) = x'(T), \end{cases} \tag{1.3}$$

where $h :]0, +\infty[\to \mathbb{R}$ is a continuously differentiable function, with a suitable singularity of repulsive type at the origin, and linear growth at $+\infty$.

In this paper, similarly as in [8,9], we consider the more general problem

$$\begin{cases} x'' + g(t, x) = sw(t) \\ x(0) = x(T), \qquad x'(0) = x'(T), \end{cases} \tag{1.4}$$

where $g : [0, T] \times]0, +\infty[\to \mathbb{R}$ satisfies some kind of Carathéodory conditions, with locally Lipschitz continuity in its second variable, and $w \in L^\infty(0, T)$. We will prove the following result.

Theorem 1.1. *Assume that:*

- *there exist $\delta > 0$ and a continuous function $f :]0, \delta] \to \mathbb{R}$ such that*

$$g(t, x) \le f(x), \quad \text{for a.e. } t \in [0, T], \text{ and every } x \in]0, \delta],$$

  and

$$\lim_{x\to 0^+} f(x) = -\infty, \qquad \int_0^\delta f(x)\, dx = -\infty;$$

- *there exist a function $a \in L^\infty(0, T)$ and a positive integer m such that*
  - *uniformly for almost every $t \in [0, T]$,*

$$\lim_{x\to+\infty} \frac{g(t, x)}{x} = a(t); \tag{1.5}$$

  - *for almost every $t \in [0, T]$,*

$$\left(\frac{m\pi}{T}\right)^2 < a_- \le a(t) \le a_+ < \left(\frac{(m+1)\pi}{T}\right)^2, \tag{1.6}$$

    for suitable real constants a_-, a_+;
  - *the unique solution $\hat{x}(t)$ to*

$$\begin{cases} x'' + a(t)x = w(t) \\ x(0) = x(T), \qquad x'(0) = x'(T) \end{cases} \tag{1.7}$$

    is strictly positive, i.e., $\hat{x}(t) > 0$ for every $t \in [0, T]$.

Then, there exists $s^* > 0$ such that, for every $s \ge s^*$, problem (1.4) has at least

$$\begin{cases} m + 2 \text{ solutions} & \text{if } m \text{ is odd}, \\ m + 1 \text{ solutions} & \text{if } m \text{ is even}. \end{cases}$$

Observe that (1.6) is a nonresonance assumption with respect to the set

$$\Sigma^D = \left\{ \left(\frac{k\pi}{T}\right)^2 \mid k = 1, 2, \ldots \right\},$$

which is the spectrum of the differential operator $x \mapsto -x''$, with Dirichlet boundary conditions on $[0, T]$. This implies that we also have nonresonance with respect to the T-periodic problem, so that the Fredholm alternative ensures the uniqueness of the solution to (1.7).

Recall that, as shown in [8, Remark 6], condition (1.6) is not enough to ensure that the solution $\hat{x}(t)$ is positive; in the case when $w(t) \equiv 1$, some sufficient conditions (in terms of some L^p-norm of $a(t)$) to guarantee this fact have been introduced in [10, Corollary 2.3].

We emphasize that, in comparison with the results obtained in [4–7], besides the introduction of a possibly nonconstant function $w(t)$, we do not assume any differentiability hypothesis on the function $g(t, x)$, and the nonresonance assumption at $+\infty$ relies only on the asymptotic behavior of the quotient $g(t, x)/x$.