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Boundary characteristic point regularity for Navier–Stokes equations: Blow-up scaling and Petrovskii-type criterion (a formal approach)

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This work is dedicated to Professor V. Lakshmikantham with great esteem

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ABSTRACT

The *three-dimensional* (3D) *Navier–Stokes equations*

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \Delta \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } Q_0, \tag{0.1}$$

where $\mathbf{u} = [u, v, w]^T$ is the vector field and p is the pressure, are considered. Here, $Q_0 \subset \mathbb{R}^3 \times [-1, 0)$ is a smooth domain of a typical *backward paraboloid* shape, with the vertex $(0, 0)$ being its only *characteristic point*: the plane $\{t = 0\}$ is tangent to ∂Q_0 at the origin, and other characteristics for $t \in [0, -1)$ intersect ∂Q_0 transversely. Dirichlet boundary conditions on the lateral boundary ∂Q_0 and smooth initial data are prescribed:

$$\mathbf{u} = 0 \quad \text{on} \quad \partial Q_0, \quad \text{and} \quad \mathbf{u}(x, -1) = \mathbf{u}_0(x) \quad \text{in} \quad Q_0 \cap \{t = -1\} \quad (\operatorname{div} \mathbf{u}_0 = 0). \tag{0.2}$$

Existence, uniqueness, and regularity studies of (0.1) in *non-cylindrical domains* were initiated in the 1960s in pioneering works by Lions, Sather, Ladyzhenskaya, and Fujita–Sauer. However, the problem of a *characteristic vertex* regularity remained open.

In this paper, the classic problem of regularity (in Wiener's sense) of the vertex $(0, 0)$ for (0.1), (0.2) is considered. Petrovskii's famous "$2\sqrt{\log \log}$-criterion" of boundary regularity for the heat equation (1934) is shown to apply. Namely, after a blow-up scaling and a special matching with a boundary layer near ∂Q_0, the regularity problem reduces to a 3D perturbed nonlinear dynamical system for the first Fourier-type coefficients of the solutions expanded using solenoidal Hermite polynomials. Finally, this confirms that the nonlinear convection term gets an exponentially decaying factor and is then negligible. Therefore, the regularity of the vertex is entirely dependent on the linear terms and hence remains the same for Stokes' and purely parabolic problems.

Well-posed Burnett equations with the minus bi-Laplacian in (0.1) are also discussed.
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1. Introduction: vertex regularity for the Navier–Stokes equations

1.1. *Navier–Stokes equations inside a non-cylindrical backward paraboloid: the first history since 1960s*

We consider the 3D Navier–Stokes equations (the 3D NSEs)

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \Delta \mathbf{u}, \qquad \operatorname{div} \mathbf{u} = 0 \quad \text{in } Q_0, \tag{1.1}$$

where $\mathbf{u} = [u, v, w]^T(x, t)$ is the vector field and $p = p(x, t)$ is the corresponding pressure.

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The NSEs (1.1) are posed in a smooth *non-cylindrical* domain

$$Q_0 \subset \mathbb{R}^3 \times [-1, 0)$$

of a typical *backward paraboloid* shape, with the vertex $(0, 0)$ being its only *characteristic point*: the plane $\{t = 0\}$ is tangent to ∂Q_0 at the origin. No characteristic points of ∂Q_0 are assumed to exist for $t \in [-1, 0)$, i.e., other characteristics $\{t = \tau\}$, for any $\tau \in [-1, 0)$, intersect ∂Q_0 transversely, in a natural sense. Next, the zero Dirichlet boundary conditions on the lateral boundary ∂Q_0 and smooth initial data at $t = -1$ are prescribed:

$$\begin{aligned} &\mathbf{u} = 0 \quad \text{on } \partial Q_0, \quad \text{and} \\ &\mathbf{u}(x, -1) = \mathbf{u}_0(x) \quad \text{in } Q_0 \cap \{t = -1\} \text{ where } \operatorname{div} \mathbf{u}_0 = 0. \end{aligned} \tag{1.2}$$

The questions of solvability, uniqueness, and regularity for the Navier–Stokes equations in *non-cylindrical* (and non-characteristic) domains, i.e., in our case, up to the vertex, for $t \le -\delta_0 < 0$, were actively studied since the 1960s. Lions began this study in 1963; see references in his classic monograph [1, Chapter 3] concerning elliptic regularization–penalization methods; as well as [2,3] (1969) as one of the first such study of weak solutions via a penalization. Another alternative, as was pointed out in [1, Chapter 3, Section 8.1], is a "rather careful using" Galerkin methods with time dependent basis functions; see [4] (1963). In 1968, Ladyzhenskaya [5] proved local existence (global for $N = 2$ and for small initial data if $N = 3$) and uniqueness of strong solutions for time-dependent domains using a different method. See [6] for more recent results, references, and other related problems.

However, the problem of regularity of a characteristic boundary point for the NSEs in any dimension $N \ge 2$ was not addressed elsewhere and remained open. Naturally, in order to proceed with regularity issues concerning the paraboloid vertex $(0, 0)$, we have to assume that a unique smooth bounded solution of (1.1), (1.2) exists in Q_0, i.e., with no L^∞-blow-up for $t < 0$.[1] In particular, as is well-known (see [5,7,8]), global smooth solutions always exist for sufficiently small initial data, so we can directly proceed with the vertex regularity, at least, for this class of solutions.

1.2. Regularity of the characteristic paraboloid vertex

Thus, the classic problem of regularity (in *Wiener's sense*, see [9]) of a boundary characteristic point for the NSE problem (1.1), (1.2) is under consideration.

Definition (*Vertex Regularity/Irregularity*)**.** According to Wiener [10], the vertex $(0, 0)$ of the given backward paraboloid Q_0 for the NSE problem (1.1), (1.2) is *regular* if, for any bounded data $\mathbf{u}_0(x)$,

$$\mathbf{u}(0, 0^-) = 0, \tag{1.3}$$

and *irregular* otherwise, i.e., at least for some initial data, (1.3) fails.

The boundary and other regularity issues for the Navier–Stokes equations in $\mathbb{R}^2$ and $\mathbb{R}^3$ have been and remain key and very popular in the modern mathematical literature, since Leray's seminal papers in 1933–34 [11,12]. Among various regularity and partial regularity results for the NSEs, the boundary regularity properties in piecewise smooth or Lipschitz domains and those with thin channels, or other non-regular domains (as we will show, such settings are key for our study) always played a special role. Mentioning Kondratiev's first study of 1967 [13], we refer to advanced results, further references, and reviews in recent papers [14–21]. See also [22,23] for the related linear Stokes problem

$$\mathbf{u}_t = -\nabla p + \Delta \mathbf{u}, \qquad \operatorname{div} \mathbf{u} = 0 \quad \text{in } Q_0, \qquad \mathbf{u}(0, x) = \mathbf{u}_0(x) \quad (\operatorname{div} \mathbf{u}_0 = 0). \tag{1.4}$$

Concerning compressible flows and other related problems, see a good survey in [16], where the 2D NSEs in a polygon domain with a convex vertex were studied.

Note that, and this is key for us in what follows, Leray in [11,12] actually posed a deep problem on both *backward* and *forward continuation* phenomena, which sound modern and advanced nowadays for general nonlinear PDE theory:

$$\begin{aligned} &\textit{Leray's blow-up scenario}\text{: self-similar blow-up} \quad \text{as } t \to T^- \ (t < T) \\ &\quad \text{and similarity collapse of this singularity as } t \to T^+ \ (t > T); \end{aligned} \tag{1.5}$$

see his precise statements and a discussion on these principal issues in [24, Section 2.2]. In this connection, such "backward blow-up scaling approaches" will be key later on.

According to our approach, we deal with a typical asymptotic problem of clarifying a generic behaviour of solutions near a "blow-up singularity" $(0, 0)$ (a "micro-scale structure" of nonlinear PDEs involved). Of course, the vertex regularity problem setting essentially and crucially depends on the *a priori* given shape of the prescribed backward paraboloids, which affects our methods of matched blow-up expansions. Anyway, we hope that our blow-up analysis on shrinking as $t \to 0^-$ subsets will eventually help to better understand the possible nature of other plausible blow-up singularities of the NSEs.

[1] But the solution is formally allowed to blow-up at the vertex $(0, 0^-)$.

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