Nonlinear Analysis 75 (2012) 4688–4698

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Blow-up of smooth solutions of the Korteweg–de Vries equation

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ARTICLE INFO

Communicated by Enzo Mitidieri

This work is dedicated to Professor V. Lakshmikantham with great esteem

Keywords:
Blow-up
KdV equation
Initial–boundary value problems
The Cauchy problem
Nonlinear capacity

ABSTRACT

The paper is devoted both to some initial–boundary value problems and to the Cauchy problem for the KdV equation.

Blow-up results are proved for these problems, by using the nonlinear capacity approach [9].

1. Introduction

The literature on Korteweg–de Vries (KdV) equation is very extensive. Roughly speaking, these contributions can be classified into two classes accordingly to methods employed.

The first direction is *algebraic*, while the second one is *analytical*.

The *algebraic theory* studies in detail the properties of solutions of the KdV equation: structure, interactions of solitons, asymptotics and so on. We refer the interested reader to [1,2] and the list of references therein for important work in this direction.

The *analytic theory* studies general properties: local and global solvability, regularity, uniqueness of various problems for the KdV equations, the Cauchy problem, and the initial–boundary value problems in bounded and unbounded spatial domains; see [3–7] and the list of references therein.

The main results are focused on regular solutions $u(t, \cdot) \in H^s$ with $s > -3/4$ and, in particular, fast decaying solutions as $|x| \to \infty$. For this, techniques based on the Fourier-type transforms are used.

The solutions growing as $|x| \to \infty$ essentially have not been studied. We point out that these kinds of singular solutions are of special interest from the viewpoint of the description, modeling phenomena of nonlinear destruction at finite time (blow-up phenomena).

The present paper is devoted to these blow-up problems.

This work is organized as follows. In Section 2 we give a short description of the method used throughout. In Section 3 we examine the initial–boundary value problem posed on bounded spatial domains. The proofs of the main results are contained in Section 4. Sections 5 and 6 are devoted to the Cauchy problem. An example of singular solution to the Cauchy problem is contained in Section 7. Finally, Section 8 contains some open problems.

To the author's knowledge, very little is known on singular solutions.

The first natural question is the following: when do the singular solutions appear, i.e., for what kinds of initial data do the solutions of the Cauchy problem blow up in finite time? When does the blow-up occur?

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doi:10.1016/j.na.2011.08.021

Clearly, if the initial data are singular, then the solution inherits this singularity. The class of such solutions is called the "singular solutions". This concept is considered in [8].

In our opinion, the problem of blow-up for *smooth* initial data is more interesting.

We point out that the first results in this direction were obtained in 1993.

Theorem 1.1 (*Bona and Saut [6]*). *Let $(x^*, t^*) \in \mathbf{R} \times \mathbf{R}_+$. Then there exists $\varphi \in L^2(\mathbf{R}) \cap L^\infty(\mathbf{R})$ such that the Cauchy problem*

$$\begin{cases} u_t + uu_x + u_{xxx} = 0, \\ u(\cdot, 0) = \varphi \end{cases}$$

has a unique solution $u \in C([0, \infty); L^2(\mathbf{R})) \cap L^2_{\mathrm{loc}}(\mathbf{R}_+; H^1_{\mathrm{loc}}(\mathbf{R}))$ which is continuous on $(\mathbf{R} \times \mathbf{R}_+) \setminus (x^*, t^*)$ and satisfies

$$\lim_{(x,t)\to(x^*,t^*)} |u(x, t)| = +\infty.$$

This result concern the so-called *dispersive blow-up*. Here the blow-up effect occurs at some isolated points due to the linear operator

$$L_0 u = u_t + u_{xxx}$$

for respective initial data.

This blow-up is generated by the linear operator L_0.

We consider blow-up from the viewpoint of the nonlinear KdV operator. Here the main factor is the nonlinear term $-uu_x$.

2. The method of investigation

Throughout this paper we shall use the method of nonlinear capacity suggested by Pohozaev in 1997 [9] and developed jointly with Mitidieri [10,11].

The essence of this method consists in the reduction of the original problem (independently of the type of the equation) to a related variational one. The extreme value of the respective functional generates the nonlinear capacity associated with the original problem. The nonlinear capacity equals the optimal a priori estimate in the respective class of test functions. Its behavior determines the existence or nonexistence of a global solution.

We note that in the applications it is not necessary to find the exact extremal value. In order to obtain sufficient conditions for blow-up, it is enough to use nearly optimal test functions.

This approach allows us to establish an homotopic invariance of the critical exponents. In [11] a Mendeleev-type table of nonlinear operators together with their critical exponents is presented.

As a simple illustration of the method of nonlinear capacity consider a very simple example of a destroying solution.

Example 2.1. Consider the Cauchy problem for the ODE

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2, \\ x(0) = c > 0. \end{cases}$$

The solution of this problem is given by

$$x(t) = \frac{1}{T - t}, \quad T = 1/c > 0.$$

Thus the solution blows up at $t = 1/c$.

Next, we analyze this problem from the point of view of nonlinear capacity.

To this end we multiply the ODE by a test function $\varphi \in C^1$ such that $\varphi \geq 0$, $\varphi(0) = 1$, $\varphi(T) = 0$. Then we obtain

$$\begin{aligned} \int_0^T x^2 \varphi \, \mathrm{d}t &= \int_0^T \varphi \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t \\ &= \varphi x \,|_0^T - \int_0^T x \varphi_t \, \mathrm{d}t = -\int_0^T x \varphi_t \, \mathrm{d}t - x_0 \\ &\leq \int_0^T x^2 \varphi \, \mathrm{d}t + \frac{1}{4} \int_0^T \frac{\varphi_t^2}{\varphi} \, \mathrm{d}t - x_0. \end{aligned}$$

Hence we get the following inequality:

$$0 \leq \frac{1}{4} \int_0^T \frac{\varphi_t^2}{\varphi} \, \mathrm{d}t - x_0. \tag{2.1}$$