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A contraction proximal point algorithm with two monotone operators

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ABSTRACT

It is a known fact that the method of alternating projections introduced long ago by von Neumann fails to converge strongly for two arbitrary nonempty, closed and convex subsets of a real Hilbert space. In this paper, a new iterative process for finding common zeros of two maximal monotone operators is introduced and strong convergence results associated with it are proved. If the two operators are subdifferentials of indicator functions, this new algorithm coincides with the old method of alternating projections. Several other important algorithms, such as the contraction proximal point algorithm, occur as special cases of our algorithm. Hence our main results generalize and unify many results that occur in the literature.

1. Introduction

Consider the following convex feasibility problem:

$$\text{find an } x \in H \quad \text{such that } x \in K_1 \cap K_2, \tag{1}$$

where K_1 and K_2 are nonempty, closed and convex subsets of a real Hilbert space H with nonempty intersection. In his 1933 paper, von Neumann showed that problem (1) can be solved by means of an iterative process. Indeed, if K_1 and K_2 are closed vector subspaces of H, von Neumann showed that any sequence (x_n) generated from the method of alternating projections

$$H \ni x_0 \mapsto x_1 = P_{K_1}x_0 \mapsto x_2 = P_{K_2}x_1 \mapsto x_3 = P_{K_1}x_2 \mapsto x_4 = P_{K_2}x_3 \mapsto \cdots,$$

converges strongly to a solution of problem (1) that is closest to the starting point x_0. The reader interested in the proof of this classical result is referred to, for example, [1,2] and the references therein. For the case when K_1 and K_2 are two arbitrary nonempty, closed and convex subsets in H with nonempty intersection, it was Bregman [3] who first showed that the sequence (x_n) generated from the method of alternating projections converges weakly to a point in $K_1 \cap K_2$. Note that in this case strong convergence fails in general, as illustrated by Hundal [4], see also [5]. In order to enforce strong convergence

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of the method of alternating resolvents, the current authors proposed several modifications of this method [6]. The most general one was given in [7], see also [8]. Such a method generates a sequence (x_n) according to the rule

$$x_{2n+1} = J^A_{\beta_n}(\alpha_n u + (1-\alpha_n)x_{2n} + e_n) \quad \text{for } n = 0, 1, \ldots, \tag{2}$$

$$x_{2n} = J^B_{\mu_n}(\lambda_n u + (1-\lambda_n)x_{2n-1} + e'_n) \quad \text{for } n = 1, 2, \ldots, \tag{3}$$

for some given $u, x_0 \in H$, where (e_n) and (e'_n) are sequences of computational errors, A and B are maximal monotone operators, and $\alpha_n, \lambda_n \in (0, 1)$ and $\beta_n, \mu_n \in (0, \infty)$. Here $J^A_\beta := (I + \beta A)^{-1}$, $\beta > 0$ (the resolvent operator of A). It was shown in [6–8] that under appropriate assumptions on the sequences of real numbers (α_n), (λ_n), (β_n) and (μ_n), and the sequences of computational errors (e_n) and (e'_n), the sequence generated from the method of alternating resolvents (2), (3) converges strongly to a point in $A^{-1}(0) \cap B^{-1}(0) =: F$ which is nearest to the point u. The method (2), (3) is in fact an extension of the method given in [9]. In addition to finding a point in F, the method proposed in [9] is capable of finding fixed points of the composition mapping $J^A_\mu J^B_\mu$, where μ is a positive real number. We observe that the method of alternating resolvents given above is defined via the regularized proximal point algorithm. Since the proximal iterates can also be generated from the contraction proximal point algorithm (CPPA),

$$x_{n+1} = \alpha_n u + \delta_n x_n + \gamma_n J^A_{\beta_n} x_n + e_n \quad \text{for } n = 0, 1, \ldots$$

(where $\alpha_n, \delta_n, \gamma_n \in (0, 1)$ with $\alpha_n + \delta_n + \gamma_n = 1$), which also produces sequences that converge strongly for a single maximal monotone operator, we shall introduce and investigate convergence properties of sequences generated from the CPPA involving two maximal monotone operators A and B. (Note that for each $n \geq 0$ and $u \in H$ fixed, the $(n+1)$th iterate defines a contraction, hence the name CPPA). More precisely, for $\alpha_n, \delta_n, \gamma_n \in (0, 1)$ with $\alpha_n + \delta_n + \gamma_n = 1$ and $\lambda_n, \rho_n, \sigma_n \in (0, 1)$ with $\lambda_n + \rho_n + \sigma_n = 1$, we introduce the following algorithm:

$$x_{2n+1} = \alpha_n u + \delta_n x_{2n} + \gamma_n J^A_{\beta_n} x_{2n} + e_n \quad \text{for } n = 0, 1, \ldots, \tag{4}$$

$$x_{2n} = \lambda_n u + \rho_n x_{2n-1} + \sigma_n J^B_{\mu_n} x_{2n-1} + e'_n \quad \text{for } n = 1, 2, \ldots, \tag{5}$$

and will prove under minimal assumptions on the sequences of parameters defining (x_n) that the sequence (x_n) converges strongly to a point in F that is nearest to u. Algorithm (4), (5) contains as special cases the inexact proximal point algorithm introduced independently by Kamimura and Takahashi [10] and Xu [11] as well as the generalized contraction proximal point algorithm which was introduced by Yao and Noor [12]. Therefore, the results of this paper extend and unify many results such as [12, Theorem 3.3], [13, Theorem 1] and [14, Theorems 2–6]. It is worth mentioning that [9, Theorem 3.3] addresses the weak convergence of the sequence generated by the method of alternating resolvents (to a point in F), whereas our present paper addresses the issue of strong convergence of the sequence given by the above modified algorithm (see (4), (5)).

2. Some preliminaries

Throughout this paper, H will be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. We recall that a map $T : H \to H$ is called nonexpansive if for every $x, y \in H$ we have $\|Tx - Ty\| \leq \|x - y\|$. The map T is called firmly nonexpansive if for every $x, y \in H$ we have

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2 .$$

It is easy to see that firmly nonexpansive mappings are nonexpansive. For more information on firmly nonexpansive mappings, we refer the reader to the excellent book by Goebel and Reich [15]. An operator $A : D(A) \subset H \to 2^H$ is said to be monotone if for every pair of points $(x, y), (x', y')$ in the graph $G(A) = \{(x, y) \in H \times H : x \in D(A), y \in Ax\}$ of A, we have $\langle x - x', y - y' \rangle \geq 0$. In other words, an operator is monotone if its graph is a monotone subset of the product space $H \times H$. An operator A is called maximal monotone if in addition to being monotone, its graph is not properly contained in the graph of any other monotone operator. Note that if A is maximal monotone, then so is its inverse A^{-1}. Given a maximal monotone operator A, one can define a single-valued and firmly nonexpansive mapping $J^A_\beta := (I + \beta A)^{-1}$ (where I is the identity operator), for every $\beta > 0$. This kind of operator is called the resolvent of A. It is known that the Yosida approximation of A, an operator defined by $A_\beta := \beta^{-1}(I - J^A_\beta)$, is maximal monotone and Lipschitzian with constant $1/\beta$ for every $\beta > 0$.

We shall use the following notation: $x_n \to x$ will mean that the sequence (x_n) converges strongly to x whereas $x_n \rightharpoonup x$ will mean that (x_n) converges weakly to x. The weak ω-limit set of a sequence (x_n) will be denoted by $\omega_w((x_n))$. That is,

$$\omega_w((x_n)) = \{x \in H : x_{n_k} \rightharpoonup x \text{ for some subsequence } (x_{n_k}) \text{ of } (x_n)\}.$$

We now recall some results which will be useful in proving our main results. We begin with an elementary property of norms in Hilbert spaces.

Lemma 1. *For all $x, y \in H$, we have*

$$\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y \rangle.$$