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Duality for ε-variational inequalities via the subdifferential calculus

Liana Cioban [a,1], Ernö Robert Csetnek [b,*]

[a] Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj-Napoca, Romania

[b] Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany

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ABSTRACT

Based on the properties of the (convex) ε-subdifferential calculus, we introduce to a general ε-variational inequality (formulated with the help of a set valued operator and a perturbation function) a dual one, expressed by making use of the (Fenchel) conjugate of the perturbation function. Under convexity hypotheses, we show that the fulfillment of a regularity condition guarantees that the primal ε-variational inequality is solvable if and only if its dual one is solvable. By particularizing the perturbation function, we obtain several dual statements and we succeed to generalize and improve a duality scheme recently given by Kum, Kim and Lee. An example justifying this generalization is also provided. Among the special instances of the general result, we rediscover also the duality scheme concerning variational inequalities due to Mosco.

1. Introduction

The issue of duality concerning variational inequalities was addressed in the literature for the first time by Mosco in the 1970s (cf. [1]). The approach uses a symmetry property of the (convex) subdifferential of a proper, convex and lower semicontinuous function defined on a real separated locally convex space (see (3) in Section 2). Since then, this problem attracted the attention of many authors and more general settings for the study of this issue were considered (let us mention here the generalizations to the vector case [2–5] and to the set-valued setting [6] or the duality concerning equilibrium problems [7–10]).

The present paper is motivated by the following duality scheme concerning ε-variational inequalities proposed by Kum et al. in [4]. For $\varepsilon \geq 0$, we consider the ε-variational inequalities:

$$\begin{array}{ll} (\mathrm{VI})_{\varepsilon}^{f,A} & \text{Find } \bar{x} \in \mathbb{R}^n \text{ for which there exists } v \in F(\bar{x}), \\ & \text{s.t. } \langle v, x-\bar{x}\rangle \geq f(A\bar{x}) - f(Ax) - \varepsilon \quad \forall x \in \mathbb{R}^n, \\ (\mathrm{DVI})_{\varepsilon}^{f,A} & \text{Find } \bar{y} \in \mathbb{R}^m \text{ for which there exists } w \in A(F^{-1}(-A^*\bar{y})), \\ & \text{s.t. } \langle w, y-\bar{y}\rangle \leq f^*(y) - f^*(\bar{y}) + \varepsilon \quad \forall y \in \mathbb{R}^m, \end{array}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set valued operator, $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function, $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping fulfilling $A^{-1}(\mathrm{dom} f) \neq \emptyset$, f^* is the (Fenchel) conjugate of f and A^* is the adjoint

* Corresponding author. Tel.: +49 371 531 39318; fax: +49 371 531 22409.
E-mail addresses: liana.cioban@ubbcluj.ro (L. Cioban), robert.csetnek@mathematik.tu-chemnitz.de (E.R. Csetnek).

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operator of A. The authors of [4] call $(\mathrm{DVI})_\varepsilon^{f,A}$ the dual variational inequality of $(\mathrm{VI})_\varepsilon^{f,A}$. It is proved in [4] that under the regularity condition $\mathrm{ri}\,\mathrm{dom} f \cap \mathrm{Im} A \neq \emptyset$ (where ri stands for the classical relative interior of a set and $\mathrm{Im} A$ is the image of the linear operator A) $(\mathrm{VI})_\varepsilon^{f,A}$ is solvable if and only if $(\mathrm{DVI})_\varepsilon^{f,A}$ is solvable. Beyond the formula for the ε-subdifferential of the function $f \circ A$, the proof relies on a general duality principle similar to the one considered by Robinson [11] in the context of composition of multifunctions (see also [12] for a duality principle for operator inclusions). Let us notice that the study of ε-variational inequalities is motivated by the fact that in many (practical) situations one is interested in finding an approximate solution of a variational inequality (see also [4, Proposition 2.1] for another aspect concerning approximation of variational inequalities).

In this paper we extend the duality scheme of Kum et al. to the infinite dimensional setting. Instead of the function $f \circ A$ in the formulation of $(\mathrm{VI})_\varepsilon^{f,A}$, we consider a general perturbation function (we refer to [13–16] for a deep study of the perturbation theory and the importance of it concerning duality in convex programming). We attach to this primal ε-variational inequality a dual one, in which the conjugate of the perturbation function is used. By using the powerful techniques of the (convex) ε-subdifferential calculus (which is well developed in the literature, see [17–19,14]) we show that in case the function involved is proper, convex and a regularity condition is fulfilled, if the primal ε-variational inequality is solvable then also its dual one is solvable. Conversely, when the dual ε-variational inequality is solvable and the function involved is proper, convex and lower semicontinuous, then also the primal one is solvable (notice that for this implication no regularity condition is needed).

We consider several particular cases of our general results and illustrate the theoretical aspects. We show that the duality scheme of Kum et al. follows as a particular instance of the main results of this paper. Moreover, we improve the results given in [4] by showing that [4, Theorem 2.1(i)] (concerning the implication $(\mathrm{VI})_\varepsilon^{f,A}$ is solvable $\Rightarrow$ $(\mathrm{DI})_\varepsilon^{f,A}$ is solvable) holds under weaker hypotheses (the lower semicontinuity of the function f is not needed and instead of the regularity condition considered in [4] we use a weaker one) and that [4, Theorem 2.1(ii)] (which addresses the implication $(\mathrm{DVI})_\varepsilon^{f,A}$ is solvable $\Rightarrow$ $(\mathrm{VI})_\varepsilon^{f,A}$ is solvable) is valid also in the absence of any regularity condition. We give also an example in order to justify the use of weaker regularity conditions than the one considered in [4]. Finally, let us mention that the duality scheme proposed by Mosco in [1] concerning variational inequalities can be seen as another particular instance of our main results.

2. Preliminaries

In this section we recall the necessary notions and results from the literature in order to make the paper as self-contained as possible. The notations are standard and follow [13–16,20,21].

Consider X a real separated locally convex space and X^* its topological dual space. We denote by $w(X^*, X)$ the weak* topology on X^* induced by X. For a nonempty set $U \subseteq X$, we denote by $\mathrm{cone}(U)$, $\mathrm{aff}(U)$, $\mathrm{lin}(U)$, $\mathrm{int}(U)$, $\mathrm{cl}(U)$, its *conical hull, affine hull, linear hull, interior*, and *closure*, respectively. If $U \subseteq X$ is a nonempty convex set, we consider its *strong quasi-relative interior* (which plays an important role in the formulation of the regularity conditions considered in this paper) defined by $\mathrm{sqri}(U) := \{x \in U : \mathrm{cone}(U - x) \text{ is a closed linear subspace of } X\}$. Let us mention that $\mathrm{int}(U) \subseteq \mathrm{sqri}(U)$ and in case X is finite dimensional we have $\mathrm{sqri}(U) = \mathrm{ri}(U)$, where $\mathrm{ri}(U)$ denotes the classical *relative interior* of U, that is the interior of U relative to $\mathrm{aff}(U)$. For more on the properties of generalized interiority notions we invite the reader to consult [14–16,22–25].

We denote by $\langle x^*, x\rangle$ (sometimes also by $\langle x, x^*\rangle$) the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. Let us consider $V \subseteq Y$ (Y being a real separated locally convex space) another nonempty set. The *projection operator* $\mathrm{pr}_U : U \times V \to U$ is defined as $\mathrm{pr}_U(u, v) = u$ for all $(u, v) \in U \times V$, while the *indicator function* of U, $\delta_U : X \to \overline{\mathbb{R}}$, is defined as $\delta_U(x) = 0$ if $x \in U$ and $+\infty$ otherwise (here $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the extended real line).

For a function $f : X \to \overline{\mathbb{R}}$ we denote by $\mathrm{dom} f = \{x \in X : f(x) < +\infty\}$ its *domain* and by $\mathrm{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}$ its *epigraph*. We call f *proper* if $\mathrm{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. The *Fenchel–Moreau conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by $f^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\}$ for all $x^* \in X^*$ and the *biconjugate* function (restricted to X) is $f^{**} : X \to \overline{\mathbb{R}}$ defined as $f^{**}(x) = \sup_{x^*\in X^*}\{\langle x^*, x\rangle - f^*(x^*)\}$ for all $x \in X$. Let us mention some properties of the conjugate function. We have the so called *Young–Fenchel inequality*: $f^*(x^*) + f(x) \ge \langle x^*, x\rangle$ for all $x \in X$ and $x^* \in X^*$. Further, $f^{**} \le f$ and according to the celebrated *Fenchel–Moreau Theorem*, if f is proper, then f is convex and lower semicontinuous if and only if $f^{**} = f$ (see [14–16,13]).

For $x \in X$ such that $f(x) \in \mathbb{R}$ we define the ε*-sudifferential* of f at x, where $\varepsilon \ge 0$, by

$$\partial_\varepsilon f(x) = \{x^* \in X^* : f(y) - f(x) \ge \langle x^*, y - x\rangle - \varepsilon\ \forall y \in X\}.$$

If $f(x) \in \{\pm\infty\}$ we take by convention $\partial_\varepsilon f(x) = \emptyset$. The set $\partial f(x) = \partial_0 f(x)$ is the classical (convex) *subdifferential* of f at x. The ε*-normal set* of U at $x \in X$ is defined by $N_U^\varepsilon(x) = \partial_\varepsilon \delta_U(x)$, that is $N_U^\varepsilon(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \le \varepsilon\ \forall y \in U\}$ when $x \in U$, and $N_U^\varepsilon(x) = \emptyset$ if $x \notin U$. The *normal cone* of U at $x \in X$ is $N_U(x) = N_U^0(x)$, that is $N_U(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \le 0\ \forall y \in U\}$, if $x \in U$ and $N_U(x) = \emptyset$ otherwise.

The following characterization of the ε-sudifferential of a proper function f at $x \in \mathrm{dom} f$ by means of conjugate functions will be useful (see [14]):

$$x^* \in \partial_\varepsilon f(x) \Leftrightarrow f(x) + f^*(x^*) \le \langle x^*, x\rangle + \varepsilon. \tag{1}$$