Nonlinear Analysis 75 (2012) 3188–3197

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Homoclinic solutions for a class of second order Hamiltonian systems with locally defined potentials

Qingye Zhang [*,1], Lipeng Chu

Department of Mathematics, Jiangxi Normal University, Nanchang 330022, PR China

ARTICLE INFO

Article history:
Received 24 August 2011
Accepted 19 December 2011
Communicated by S. Ahmad

MSC:
34J45
35J61
58E30

Keywords:
Hamiltonian system
Homoclinic solution
Variational method

ABSTRACT

In this paper, we study the existence of infinitely many homoclinic solutions for a class of second order Hamiltonian systems $\ddot{u} - L(t)u + W_u(t, u) = 0$, $\forall t \in \mathbb{R}$, where $L(t)$ is unnecessarily positive definite for all $t \in \mathbb{R}$, and $W(t, u)$ is only locally defined near the origin with respect to u and not assumed to be periodic with respect to t.

1. Introduction and main results

Consider the following second order Hamiltonian system

$$\ddot{u} - L(t)u + W_u(t, u) = 0, \quad \forall t \in \mathbb{R}, \tag{HS}$$

where $u = (u_1, \ldots, u_N) \in \mathbb{R}^N$, $L \in C\left(\mathbb{R}, \mathbb{R}^{N^2}\right)$ is a symmetric matrix-valued function, and $W_u(t, u)$ denotes the gradient of $W(t, u)$ with respect to u. Here, as usual, we say that a solution u of (HS) is homoclinic (to 0) if $u \in C^2\left(\mathbb{R}, \mathbb{R}^N\right)$, $u \neq 0$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$.

During the last several decades, the existence and multiplicity of homoclinic solutions for (HS) have been extensively investigated in many papers (see, e.g., [1–24] and the references therein) via the variational methods. Most of them (see [1–3,6,8–10,13–15]) treated the case where $L(t)$ and $W(t, u)$ are either independent of t or periodic in t. In this kind of problem, the function $L(t)$ plays an important role. If $L(t)$ is neither a constant nor periodic, the problem is quite different from the ones just described, because of the lack of compactness of the Sobolev embedding. After the work of Rabinowitz and Tanaka [15], many results (see, e.g., [4,5,7,9,11,12,16–24]) were obtained for the case where $L(t)$ is neither a constant nor periodic. Among them, except for [7,12,15,17,18,20–22,24] all known results were obtained under the assumption that $L(t)$ is positive definite for all $t \in \mathbb{R}$, that is,

$$L(t)u \cdot u > 0, \quad \forall t \in \mathbb{R} \text{ and } u \in \mathbb{R}^N \setminus \{0\},$$

* Corresponding author. Tel.: +86 791 8120367; fax: +86 791 8120367.
E-mail address: qingyezhang@gmail.com (Q. Zhang).

1 The first author was supported by Tianyuan Fund for Mathematics (11126146).

0362-546X/$ – see front matter
doi:10.1016/j.na.2011.12.018

where $\cdot$ denotes the standard inner product $\mathbb{R}^N$. Besides, we emphasize that in all these papers $W(t, u)$ was always required to satisfy some kind of growth conditions at infinity with respect to u, such as superquadratic, asymptotically quadratic or subquadratic growth.

In the present paper, we will study the existence of infinitely many homoclinic solutions for (HS) in the case where $L(t)$ is coercive but unnecessarily positive definite for all $t \in \mathbb{R}$, and $W(t, u)$ is only locally defined near the origin with respect to u. More precisely, we present the following assumptions:

(L_1) There is a constant $\alpha < 2$ such that $l(t)|t|^{\alpha-2} \to \infty$ as $|t| \to \infty$, where $l(t)$ is the smallest eigenvalue of $L(t)$, i.e.,

$$l(t) \equiv \inf_{\xi \in \mathbb{R}^N,\, |\xi|=1} L(t)\xi \cdot \xi.$$

(L_2) There exist constants $a > 0$ and $r > 0$ such that
(i) $L \in C^1(\mathbb{R}, \mathbb{R}^{N^2})$ and $|L'(t)u| \leq a|L(t)u|,\ \forall\, |t| \geq r$ and $u \in \mathbb{R}^N$, or
(ii) $L \in C^2(\mathbb{R}, \mathbb{R}^{N^2})$ and $\langle (L''(t) - aL(t))u, u\rangle \leq 0,\ \forall\, |t| \geq r$ and $u \in \mathbb{R}^N$,
where $L'(t) = (d/dt)L(t)$ and $L''(t) = (d^2/dt^2)L(t)$,

(W_1) $W \in C^1(\mathbb{R} \times B_\delta(0), \mathbb{R})$ is even in u and $W(t, 0) \equiv 0$, where $B_\delta(0)$ is the ball in $\mathbb{R}^N$ centered at 0 with radius $\delta > 0$;

(W_2) There exist constants $c_1 > 0$ and $\frac{1}{2} \leq \nu \in (1/(3-\alpha), 1)$ such that

$$|W_u(t, u)| \leq c_1|u|^\nu, \quad \forall\, (t, u) \in \mathbb{R} \times B_\delta(0);$$

(W_3)

$$\lim_{|u|\to 0} \frac{W(t, u)}{|u|^2} = \infty \quad \text{uniformly for } t \in \mathbb{R};$$

(W_4) $2W(t, u) - W_u(t, u) \cdot u > 0$ for all $t \in \mathbb{R}$ and $u \neq 0$.

Our main result reads as follows.

Theorem 1.1. *Suppose that* (L_1), (L_2) *and* (W_1)–(W_4) *are satisfied. Then* (HS) *possesses a sequence of homoclinic solutions* $\{u_n\}$ *such that* $\max_{t\in\mathbb{R}} |u_n(t)| \to 0$ *as* $n \to \infty$.

Remark 1.2. In our Theorem 1.1, there is no assumption on W for u large enough and hence the potential is only locally defined near the origin with respect to u. This is in sharp contrast with the aforementioned references. To the best of our knowledge, there is little literature concerning the existence of infinitely many homoclinic solutions for (HS) in this situation. In addition, our Theorem 1.1 is in some sense an improvement for some related results in the existing literature. For instance, it is easy to check that L and W will satisfy (L_1), (L_2) and (W_1)–(W_4) in our Theorem 1.1 provided that they satisfy all the conditions required in Theorem 1.2 in [7].

Following partially the idea of [25] in dealing with the Dirichlet boundary problems, we will first modify $W(t, u)$ for u outside a neighborhood of the origin 0 to get $\widetilde{W}(t, u)$ and introduce the following modified Hamiltonian system

$$\ddot{u} - L(t)u + \widetilde{W}_u(t, u) = 0, \quad \forall\, t \in \mathbb{R}, \tag{$\widetilde{\text{HS}}$}$$

where $\widetilde{W}$ will be specified in Section 2. Then we show by variational methods that Hamiltonian system ($\widetilde{\text{HS}}$) possesses a sequence of homoclinic solutions, which converges to zero in L^∞ norm. Consequently, we obtain infinitely many homoclinic solutions for the original Hamiltonian system (HS).

2. Variational setting and proof of the main result

In order to prove our main result via the critical point theory, we first establish the variational setting for ($\widetilde{\text{HS}}$).

In what follows it will always be assumed that (L_1) and (L_2) are satisfied. Denote by A the self-adjoint extension of the operator $-\frac{d^2}{dt^2} + L(t)$ with domain $\mathscr{D}(A) \subset L^2 \equiv L^2(\mathbb{R}, \mathbb{R}^N)$. Let $\{E(\lambda) : -\infty < \lambda < \infty\}$ and $|A|$ be the spectral resolution and the absolute value of A respectively, and $|A|^{1/2}$ be the square root of $|A|$ with domain $\mathscr{D}(|A|)$. Set $U = I - E(0) - E(-0)$, where I is the identity map on L^2. Then U commutes with A, $|A|$ and $|A|^{1/2}$, and $A = U|A|$ is the polar decomposition of A (see [26]). Let $E = \mathscr{D}(|A|^{1/2})$, and define on E the inner product and norm by

$$(u, v)_0 = \left(|A|^{1/2}u, |A|^{1/2}v\right)_2 + (u, v)_2,$$

$$\|u\|_0 = (u, u)_0^{1/2},$$

where $(\cdot, \cdot)_2$ denotes the usual inner product on L^2. Then E is a Hilbert space.

By Lemmas 2.2 and 2.3 in [7], we have the following two lemmas.

Lemma 2.1. *If L satisfies* (L_1), *then E is compactly embedded into* $L^p \equiv L^p(\mathbb{R}, \mathbb{R}^N)$ *for all* $1 \leq p \in (2/(3-\alpha), \infty]$.