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A boundary value problem for fractional differential equation with p-Laplacian operator at resonance[☆]

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A B S T R A C T

In this paper, by using the coincidence degree theory, we consider the following boundary value problem for fractional p-Laplacian equation

$$\begin{cases} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}x(t)) = f(t, x(t), D_{0+}^{\alpha}x(t)), t \in [0, 1], \\ D_{0+}^{\alpha}x(0) = D_{0+}^{\alpha}x(1) = 0, \end{cases}$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, D_{0+}^{α} is a Caputo fractional derivative, and $p > 1$, $\phi_p(s) = |s|^{p-2}s$ is a p-Laplacian operator. A new result on the existence of solutions for the above fractional boundary value problem is obtained, which generalize and enrich some known results to some extent from the literature.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration on an arbitrary order that can be noninteger. This subject, as old as the problem of ordinary differential calculus, can go back to the times when Leibniz and Newton invented differential calculus. As is known to all, the problem for fractional derivative was originally raised by Leibniz in a letter, dated September 30, 1695. A fractional derivative arises from many physical processes, such as a non-Markovian diffusion process with memory (see [1]), charge transport in amorphous semiconductors (see [2]), propagations of mechanical waves in viscoelastic media (see [3]), etc. Moreover, phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are also described by differential equations of fractional order (see [4–8]). For instance, Pereira et al. (see [9]) considered the following fractional Van der Pol equation

$$D^{\lambda}x(t) + \alpha(x^2(t) - 1)x'(t) + x(t) = 0, \quad 1 < \lambda < 2, \tag{1.1}$$

where D^{λ} is the fractional derivative of order λ and α is a control parameter that reflects the degree of nonlinearity of the system. Eq. (1.1) is obtained by substituting the capacitance by a fractance in the nonlinear RLC circuit model.

Recently, fractional differential equations have been of great interest due to the intensive development of theory of fractional calculus itself and its applications. For example, for fractional initial value problems, the existence and multiplicity

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of solutions (or positive solutions) were discussed in [10–13]. On the other hand, for fractional boundary value problems at nonresonance, Agarwal et al. (see [14]) considered a two-point boundary value problem given by

$$\begin{cases} D^{\alpha}x(t) + f(t, x(t), D^{\mu}x(t)) = 0, \\ x(0) = x(1) = 0, \end{cases}$$

where $1 < \alpha < 2$, $\mu > 0$ are real numbers, $\alpha - \mu \geq 1$, and D^{α} is the Riemann–Liouville fractional derivative. For fractional boundary value problems at resonance, Bai (see [15]) considered a m-point boundary value problem in the form

$$\begin{cases} D_{0+}^{\alpha}x(t) = f(t, x(t), D_{0+}^{\alpha-1}x(t)) + e(t), & 0 < t < 1, \\ I_{0+}^{2-\alpha}x(t)|_{t=0} = 0, D_{0+}^{\alpha-1}x(1) = \sum_{i=1}^{m-2} \beta_i D_{0+}^{\alpha-1}x(\eta_i), \end{cases}$$

where $1 < \alpha \leq 2$ is a real number, $\eta_i \in (0, 1)$, $\beta_i \in \mathbb{R}$ are given constants such that $\sum_{i=1}^{m-2} \beta_i = 1$, and D_{0+}^{α}, I_{0+}^{α} are the Riemann–Liouville differentiation and integration. For more papers on fractional boundary value problems, see [16–20] and the references therein.

The turbulent flow in a porous medium is a fundamental mechanics problem. For studying this type of problems, Leibenson (see [21]) introduced the p-Laplacian equation as follows

$$(\phi_p(x'(t)))' = f(t, x(t), x'(t)), \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant such that $\frac{1}{p} + \frac{1}{q} = 1$.

In the past few decades, many important results relative to Eq. (1.2) with certain boundary value conditions had been obtained. We refer the reader to [22–26] and the references cited therein. However, to the best of our knowledge, there are relatively few results on boundary value problems for fractional p-Laplacian equations.

Motivated by the work above, in this paper, we investigate the existence of solutions for the boundary value problem (BVP for short) of fractional p-Laplacian equation with the following form

$$\begin{cases} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}x(t)) = f(t, x(t), D_{0+}^{\alpha}x(t)), & t \in [0, 1], \\ D_{0+}^{\alpha}x(0) = D_{0+}^{\alpha}x(1) = 0, \end{cases} \tag{1.3}$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, D_{0+}^{α} is Caputo fractional derivative, and $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

Note that, the nonlinear operator $D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha})$ reduces to the linear operator $D_{0+}^{\beta}D_{0+}^{\alpha}$ when $p = 2$ and the additive index law

$$D_{0+}^{\beta}D_{0+}^{\alpha}u(t) = D_{0+}^{\alpha+\beta}u(t)$$

holds under some reasonable constraints on the function $u(t)$ (see [27]). Moreover, BVP (1.3) happens to be at resonance in the sense that its associated linear homogeneous boundary value problem

$$\begin{cases} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}x(t)) = 0, & t \in [0, 1], \\ D_{0+}^{\alpha}x(0) = D_{0+}^{\alpha}x(1) = 0 \end{cases}$$

has a nontrivial solution $x(t) = c$, where $c \in \mathbb{R}$.

The rest of this paper is organized as follows. Section 2 contains some necessary notations, definitions and lemmas. In Section 3, basing on the coincidence degree theory of Mawhin (see [28]), we establish a theorem on existence of solutions for BVP (1.3) under nonlinear growth restriction of f. Finally, in Section 4, an example is given to illustrate the main result. Our result is different from those of bibliographies listed above.

2. Preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions about fractional calculus theory, which can be found, for instance, in [29,30].

Definition 2.1. The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}u(s)ds,$$

provided that the right side integral is pointwise defined on $(0, +\infty)$.