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On the blow up scenario for a class of parabolic moving boundary problems

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ABSTRACT

We consider maximally continued classical solutions of a large class of parabolic moving boundary problems. If the maximal existence time is finite, we describe the blow up mechanism: either a suitable norm of the bulk density blows up or the geometry of the interface collapses. This can also be seen as a sufficient condition for global in time existence of classical solutions. Moreover, we prove a representation theorem saying, that any closed compact connected hypersurface of Hölder regularity class $c^{k,\alpha}$ can be regarded as a graph over an analytic hypersurface, provided $k \geq 2$.

1. Introduction

Functional analytic methods as the theory of analytic semigroups or the concept of maximal regularity have proven to be an efficient tool in order to study analytic properties of solutions of moving boundary problems. Nowadays, it is not only possible to prove the local in time existence of classical solutions for many moving boundary problems, but also to observe strong effects of regularization and to investigate the problem of stability/instability of equilibria.[1] However, necessary for this approach is always the usage of a suitable coordinate transformation, say the Hanzawa-diffeomorphism. This construction does not allow a classical solution to leave a prescribed small neighborhood of the initial surface. Thus, many types of dynamical behavior (the formation of singularities for example) have not yet been investigated. In this paper we shall make a first step in this direction. We focus on maximally continued classical solutions of a large class of moving boundary problems. We generalize the usual statement for nonlinear parabolic PDEs on a fixed domain, claiming, that classical solutions can be continued until a certain blow-up occurs. As mentioned above, in the case of a moving and a priori unknown domain, blow up can also mean the formation of a singularity on the boundary manifold. The main result of this paper is the following (Theorem 5.2): if the maximal existence time is finite, a suitable norm of the bulk density must blow up or the geometry collapses. More precisely, we prove that any classical solution with the property that

- the surfaces satisfy a uniform ball condition;
- a suitable norm of the bulk density (strictly weaker compared to the space of initial data) remains bounded

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[1] We emphasize that in this paper we are not interested in mere existence results, but we aim to refer the reader to [1–8] where such issues have been discussed.

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must exist globally in time. This has already been done for a problem modeling avascular tumor growth in [2]; cf. Section 6.1 of this paper. We shall considerably generalize this idea here (Theorem 5.2, Lemmas 5.7 and 5.8): we show that the method used in [2] applies to a whole class of parabolic moving boundary problems.

As mentioned above, a possible treatment of various free boundary problems is based on the usage of a suitable coordinate transformation giving an equivalent formulation of the problem consisting in a coupled system of nonlinear (evolution) equations involving a fixed reference domain only, see [9–13]. A widely used transformation is the so called Hanzawa-diffeomorphism: The moving domain is considered as the graph of a function which measures the signed distance to some fixed and smooth reference manifold ([14,8]). The assumption that a given initial surface can be represented that way is necessary for this approach. We shall prove a suitable representation theorem here (Theorem 4.2) containing an optimal regularity result. To the best of our knowledge, a proof of the statement of this representation theorem has never been published, although it is a frequently made assumption in the context of free boundary problems.

Observe that one advantage of the approach using the Hanzawa diffeomorphism (also called the direct mapping method) is the possibility precisely to measure the (in particular spatial) regularity of the moving surface. But in general the evolving surface will loose the property of being a graph over some reference domain approximating the initial surface. Thus, even the precise definition of a maximally continued classical solution is not at all obvious. We shall make use of Theorem 4.2 for this definition.

This paper is organized as follows: Some basic notation and conventions are introduced in Section 2. In Section 3 we shall capture some paradigmatic examples by a general notation. These and other examples are discussed in detail in Section 6. In particular, for each example we specify the norm of the bulk density which has to be controlled in order to establish global existence (the space F in the notation of Section 4). It turns out that in many cases the control of the gradient at the moving boundary is essential.

The abstract setting is made precise in Sections 4 and 5 is devoted to state our main result. Here, Theorem 5.2 is the most general formulation, while the Lemmas 5.7 and 5.8 treat special situations. In particular, they offer strategies how to verify the assumptions of Theorem 5.2.

Note that all the examples in Sections 6.1–6.4 are so called one phase problems: they consist of a set of differential equations given only inside the unknown boundary manifold, and they are coupled to it via boundary conditions, of course. We shall explain the natural generalization of our result to the case of two phases in Section 6.5. Finally, Section 7 gives the necessary proofs.

2. Notations

Throughout this article we fix the numbers $n, j, k, m \in \mathbb{N}$, $n \geq 2$, $k \geq 3$, and $\alpha \in (0, 1)$. Moreover, we denote by $\mathcal{DOM}(k, \alpha)$ the set of all $c^{k,\alpha}$- domains in $\mathbb{R}^n$, that is the set of all bounded connected open subsets $D \subset \mathbb{R}^n$ such that ∂D is a compact closed hypersurface of regularity class $c^{k,\alpha}$. Here, $c^{k,\alpha}$ denotes the little Hölder space, i.e. the closure of the (sufficiently) smooth functions in the usual Hölder norm $\|\cdot\|_{k,\alpha}$. If X and Y are Banach spaces and $M \subset X$, $B(M, Y)$ denotes the bounded, and $BUC(M, Y)$ [$BUC^{\mu}(M, Y)$, if M is open] denotes the bounded and uniformly continuous mappings [possessing bounded and uniformly continuous derivatives up to order $\mu \in \mathbb{N}$] from X to Y. Throughout the following we fix the functions

- $G \in C^{\infty}(\mathbb{R}^m, \mathbb{R}^m)$;
- $(B, E) \in C^{\infty}(\mathbb{R}^{(2m+1)}, \mathbb{R}^m \times \mathbb{R})$;
- $\Phi \in C^{\infty}(\mathbb{R}^n, \mathbb{R}^m)$;

and use the notation $G = (G_1, \dots, G_m)$, $\Phi = (\Phi_1, \dots, \Phi_m)$, $B = (B_1, \dots, B_m)$.

3. One phase problems

In this Section we consider an abstract dynamical system

$$\begin{cases} \mathcal{E}\dot{u} + Au = G(u) & \text{in } \Omega(t) \\ B(u, \partial_\nu u, H) = \Phi & \text{on } \Gamma(t) \\ V = E(u, \partial_\nu u, H) & \text{on } \Gamma(t), \end{cases} \tag{3.1}$$

where $t > 0$, $u = (u_1, \dots, u_m)$ and

$$\mathcal{E} := \begin{pmatrix} \mathbf{E^j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{3.2}$$

Here, $\mathbf{E^j}$ denotes the identity map in $\mathbb{R}^j$. By A we denote an m- vector of linear (elliptic) second order differential operators acting on u as follows:

$$(A_1, \dots, A_m)(u_1, \dots, u_m) := (A_1 u_1, \dots, A_m u_m).$$

Finally, $\{\Omega(t)\}$ is a family of time dependent unknown domains, and $\partial_\nu = \partial_{\nu(t)}$ is the derivate in direction of the outer unit normal field with respect to $\Gamma(t) := \partial\Omega(t)$, acting on u simply componentwise.