Nonlinear Analysis 75 (2012) 3994–4005

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On the one-dimensional p-Laplacian with a singular nonlinearity[☆]

Wenshu Zhou[a], Xulong Qin[b], Guokai Xu[c], Xiaodan Wei[a,*]

[a] Department of Mathematics, Dalian Nationalities University, Dalian 116600, China
[b] Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, China
[c] College of Electromechanical and Information Engineering, Dalian Nationalities University, Dalian 116600, China

ARTICLE INFO

Article history:
Received 2 July 2011
Accepted 17 February 2012
Communicated by S. Carl

MSC:
34B18

Keywords:
p-Laplacian
Singularity
Positive solution
Existence
Nonexistence

ABSTRACT

In this paper, we are concerned with the existence and nonexistence of positive solutions of boundary value problems for the one-dimensional p-Laplacian with a singular nonlinearity. In the case of the model equation, we give the necessary and sufficient conditions of the existence of positive solutions for both the Dirichlet problem and the periodic problem.

1. Introduction

In this paper, we are concerned with the existence and nonexistence of positive solutions for the singular p-Laplacian equation:

$$\left[\phi_p(u')\right]' - h(t,u)|u'|^p + f(t,u,u') = 0, \quad 0 < t < 1, \tag{1.1}$$

with either the Dirichlet boundary conditions

$$u(1) = u(0) = 0, \tag{1.2}$$

or the periodic boundary conditions

$$u'(1) = u'(0) = u(1) = u(0) = 0, \tag{1.3}$$

where $\phi_p(s) = |s|^{p-2}s$ with $p > 1$, $h(t,z)$ is nonnegative and continuous in $[0,1] \times (\mathbb{R} \setminus \{0\})$ and may be singular at $z = 0$, and $f(t,z,r)$ is nonnegative and continuous on $[0,1] \times \mathbb{R} \times \mathbb{R}$. The model equation is

$$\left[\phi_p(u')\right]' - \lambda \frac{|u'|^p}{u^m} + f(t,u,u') = 0, \quad 0 < t < 1, \tag{1.4}$$

where $\lambda, m > 0$.

[☆] This work is supported by NNSF of China (grant nos. 10901030, 11071100, 11001278), Key Technology in the National Research and Development Pillar Program (no. 2009BAH41B05), China Postdoctoral Science Foundation (no. 20090450191), and Dalian Nationalities University (no. DC110109).
* Corresponding author. Tel.: +86 411 87189428; fax: +86 411 87189428.
E-mail address: weixiaodancat@126.com (X. Wei).

0362-546X/$ – see front matter
doi:10.1016/j.na.2012.02.015

Besides singularity, another main feature of (1.1) is that the lower term has p-growth with respect to the first order derivative.

Eq. (1.4) is strongly connected to the degenerate parabolic equation

$$u_t = u[\phi_p(u_x)]_x - \lambda|u_x|^p, \tag{1.5}$$

where $\lambda > 0$ is a constant, which appears in the investigation of groundwater flow in a water-absorbing fissurized porous rock and non-Newtonian filtration [1–4]. Indeed, if the solution u of (1.5) is of the form $u = t^{-1/(p-1)}w(x)$ with $w > 0$, then w must satisfy (in some sense)

$$[\phi_p(w')]' - \lambda\frac{|w'|^p}{w} + \frac{1}{p-1} = 0.$$

The existence of positive solutions for (1.4) with different boundary conditions has recently been studied in [2,5–8]. In [2,5,6], the equations considered may have time singularities. In the present paper, we extend some existence results in [5,7,8] and obtain some sufficient conditions on the nonexistence of positive solutions. As by-products of our main results, we prove, under some auxiliary assumptions on f, that the condition $m < p$ is necessary and sufficient for the existence of positive solutions to problem (1.4)–(1.2), and the condition $1 \leqslant m < p$ is necessary and sufficient for the existence of positive solutions to problem (1.4)–(1.3). Our arguments on the existence are based on regularization technique, while in the arguments on the nonexistence, Gronwall's inequality, embedding inequality and a scaling technique will play important roles.

In the case where $h \equiv 0$, (1.1) becomes

$$\left[\phi_p(u')\right]' + f(t, u, u') = 0, \quad 0 < t < 1. \tag{1.6}$$

Recently, problem (1.6)–(1.2) have been studied extensively, see for example [9–27], in which $f(t, z, r)$ may change sign and be singular at $t = 0$, $t = 1$ and $z = 0$. Some basic results were obtained in those papers. We point out that because the singularity at $u = 0$ and the p-growth with respect to u' as a whole appear in (1.1), it is not in considerations of those papers.

By a solution u of problem (1.1)–(1.2) we mean that $u \in \mathscr{W}$, where

$$\mathscr{W} := \{w \in C^1[0, 1];\ w > 0 \text{ in } (0, 1),\ \phi_p(w') \in C^1(0, 1)\},$$

and u satisfies (1.1) and (1.2). If u is a solution problem (1.1)–(1.2) and satisfies (1.3), then we call it is a solution for problem (1.1)–(1.3).

To study the existence and nonexistence of solutions, we impose some additional conditions on h and f.

We first study the existence of solutions. For h, we suppose that

$$0 \leqslant h(t, z) \leqslant g(z), \quad \forall t \in [0, 1],\ \forall z > 0, \tag{1.7}$$

where $g : (0, +\infty) \to [0, +\infty)$ is continuous in $(0, +\infty)$ and differentiable in $(0, \xi)$ for some $\xi > 0$, such that

$$g' \leqslant 0 \text{ in } (0, \xi), \quad \lim_{s\to 0^+} g(s) = +\infty, \tag{1.8}$$

$$\lim_{s\to 0^+} \int_s^1 [g(z)]^{1/p} dz < +\infty. \tag{1.9}$$

For f, we suppose that there exist constants $\beta \geqslant \alpha > 0$, $\gamma > 0$ such that

$$\alpha \leqslant f(t, z, r) \leqslant \beta + \gamma|r|^{p-1}, \quad \forall (t, z, r) \in [0, 1] \times [0, +\infty) \times \mathbb{R}. \tag{1.10}$$

We obtain

Theorem 1.1. *Let (1.7)–(1.10) be satisfied. Then there exists a solution for problem (1.1)–(1.2).*

Theorem 1.2. *Let $h(t, z) \equiv g(z)$ satisfy (1.8) and (1.9), and suppose (1.10) holds. Then there exists a solution for problem (1.1)–(1.3) if and only if $\lim_{s\to 0^+} \int_s^1 g(z)dz = +\infty$.*

Remark 1.1. Some examples of the functions satisfying (1.8)–(1.9) are:

(1) $g(z) = \lambda/z^m$, where $\lambda > 0$ and $0 < m < p$;
(2) $g(z) = (\ln z)^2$;
(3) $g(z) = 1/(e^z - 1)$;
(4) $g(z) = (\ln z)^2$ if $0 < z < \epsilon$; $g(z) = (\ln \epsilon)^2$ if $z \geqslant \epsilon$, where $\epsilon > 0$.