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Positive ground state solutions for the critical Klein–Gordon–Maxwell system with potentials

Paulo C. Carrião [a], Patrícia L. Cunha [b], Olímpio H. Miyagaki [c,*]

[a] *Departamento Matemática, UFMG, Belo Horizonte-MG, Brazil*

[b] *Departamento Matemática, UFSCar, São Carlos-SP, Brazil*

[c] *Departamento Matemática, UFJF, Juiz de Fora-MG, Brazil*

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ABSTRACT

This paper deals with the Klein–Gordon–Maxwell system when the nonlinearity exhibits critical growth. We prove the existence of positive ground state solutions for this system when a periodic potential V is introduced. The method combines the minimization of the corresponding Euler–Lagrange functional on the Nehari manifold with the Brézis and Nirenberg technique.

1. Introduction

In this paper, we consider the Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = \mu |u|^{q-2}u + |u|^{2^*-2}u & \text{in } \mathbb{R}^3 \\ \Delta \phi = (\omega + \phi)u^2 & \text{in } \mathbb{R}^3 \end{cases} \qquad (\mathcal{KGM})$$

where μ and ω are positive real constants, $2 < q < 2^* = 6$ and $u, \phi : \mathbb{R}^3 \to \mathbb{R}$. Moreover, we assume the following hypothesis on the continuous function V:

(V1) $V(x+p) = V(x),\ x \in \mathbb{R}^3,\ p \in \mathbb{Z}^3$
(V2) There exists $V_0 > 0$ such that $V(x) \geq V_0 > 0, x \in \mathbb{R}^3$,
 where $V_0 > \frac{2(4-q)}{q-2}\omega^2$ if $2 < q < 4$.

This system appears as a model which describes the nonlinear Klein–Gordon field interacting with the electromagnetic field in the electrostatic case. The unknowns of the system are the field u associated to the particle and the electric potential ϕ. The presence of the nonlinear term simulates the interaction between many particles or external nonlinear perturbations.

Let us recall some previous results that led us to the present research.

* Corresponding author. Tel.: +55 31 38917070.
E-mail addresses: carrion@mat.ufmg.br (P.C. Carrião), patcunha80@gmail.com (P.L. Cunha), ohmiyagaki@gmail.com (O.H. Miyagaki).

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The first result is due to Benci and Fortunato. In [1], they proved the existence of infinitely many radially symmetric solutions for the Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + [m_0^2 - (\omega + \phi)^2]u = |u|^{q-2}u & \text{in } \mathbb{R}^3 \\ \Delta \phi = (\omega + \phi)u^2 & \text{in } \mathbb{R}^3 \end{cases} \tag{1}$$

considering subcritical behavior on the nonlinearity under the assumptions $|m_0| > |\omega|$ and $4 < q < 6$. In [2], D'Aprile and Mugnai covered the case $2 < q < 4$ assuming $m_0\sqrt{p-2} > \sqrt{2}\omega > 0$ and the case $q = 4$ assuming $m_0 > \omega > 0$.

Motivated by the approach of Benci and Fortunato, Cassani [3] considered system (1) for the critical case by adding a lower order perturbation:

$$\begin{cases} -\Delta u + [m_0^2 - (\omega + \phi)^2]u = \mu|u|^{q-2}u + |u|^{2^*-2}u & \text{in } \mathbb{R}^3 \\ \Delta \phi = (\omega + \phi)u^2 & \text{in } \mathbb{R}^3 \end{cases} \tag{2}$$

where $\mu > 0$. He was able to show that

(i) if $|m_0| > |\omega|$ and $4 < q < 2^*$, then for each $\mu > 0$, there exists a radially symmetric solution for system (2);
(ii) if $|m_0| > |\omega|$ and $q = 4$, then system (2) has a radially symmetric solution provided that μ is sufficiently large.

The class of $(\mathcal{KGM})$ system presented in this paper with such potential $V(x)$ is closely related to a number of several other works. In fact, the potential $V(x)$ also satisfies the constant case $m_0^2 - \omega^2$ which has been extensively considered; see e.g. [4,5,1,3,2,6].

In [7], Georgiev and Visciglia also introduced a class of $(\mathcal{KGM})$ system with potentials; however they considered a small external Coulomb potential in the corresponding Lagrangian density.

We observe that without loss of generality we may assume $\omega > 0$, because if (u, ϕ) is a solution of $(\mathcal{KGM})$ system, then $(u, -\phi)$ will be a solution corresponding to $-\omega$. Therefore, the sign of ω is not essential when looking for existence of solutions.

The investigation of *ground state* solutions, that is, couples (u, ϕ) which solve $(\mathcal{KGM})$ and minimize the action functional associated to $(\mathcal{KGM})$ among all possible nontrivial solutions, has been considered by many authors in a plethora of problems. See, for example, [5,8–11].

The authors Azzollini and Pomponio [5] established the existence of ground state solutions for the subcritical Klein–Gordon–Maxwell system (1), under the following assumptions:

(i) $4 \le q < 6$ and $m_0 > \omega$;
(ii) $2 < q < 4$ and $m_0\sqrt{q-2} > \omega\sqrt{6-q}$.

Their technique consisted in minimizing the corresponding functional of (1) on the Nehari manifold.

In the present paper, we go one step further and extend Theorem 1.1 in [5] for the critical growth case. Moreover, we establish the sign of the solution.

Our main result is as follows.

Theorem 1.1. *If conditions* (V1) *and* (V2) *hold, then* $(\mathcal{KGM})$ *system has a positive ground state solution for each* $\mu > 0$ *if* $4 < q < 6$ *and for* μ *sufficiently large if* $2 < q \le 4$.

Our approach combines the minimization of the corresponding functional of $(\mathcal{KGM})$ system on the Nehari manifold with the Brézis and Nirenberg technique.

2. Variational setting

In this section, we introduce notations and prove some preliminary results concerning the variational structure for $(\mathcal{KGM})$ system.

Throughout this paper, C and C_i are positive constants which may change from line to line.

Let us consider the Sobolev space E endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\, dx$$

which is equivalent to the usual Sobolev norm on $H^1(\mathbb{R}^3)$. Also $\mathcal{D}^{1,2} \equiv \mathcal{D}^{1,2}(\mathbb{R}^3)$ represents the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|^2_{\mathcal{D}^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2\, dx.$$

For any $1 \le s < \infty$, $L^s(\mathbb{R}^3)$ is the usual Lebesgue space endowed with the norm

$$\|u\|_s^s = \int_{\mathbb{R}^3} |u|^s\, dx.$$