Nonlinear Analysis 75 (2012) 4137–4144

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Periodic solutions of Rayleigh equations via time-maps ☆

Tiantian Ma *

School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China

ARTICLE INFO

Article history:
Received 31 December 2011
Accepted 4 March 2012
Communicated by S. Ahmad

MSC:
34C25
34B15

Keywords:
Rayleigh equation
Periodic solution
Continuation lemma

ABSTRACT

In this paper, we study the existence of periodic solutions of Rayleigh equations

$$x'' + f(t, x') + g(x) = e(t),$$

where $f : \mathrm{R}^2 \to \mathrm{R}$ is continuous and T-periodic with respect to the first variable, $g, e : \mathrm{R} \to \mathrm{R}$ are continuous and e is T-periodic. We prove that the given equation possesses at least one T-periodic solution provided that either

$$\limsup_{c\to+\infty} \tau(c) + \liminf_{c\to-\infty} \tau(c) > T$$

or

$$\liminf_{c\to+\infty} \tau(c) + \limsup_{c\to-\infty} \tau(c) > T$$

is satisfied, where τ is the time-map defined in Section 1.

1. Introduction

In this paper, we are concerned with the existence of periodic solutions of Rayleigh equations

$$x'' + f(t, x') + g(x) = e(t), \tag{1.1}$$

where $f : R^2 \to R, g, e : R \to R$ are continuous and f is T-periodic with respect to the first variable, e is T-periodic.

Eq. (1.1) has wide applications in physics, mechanics, engineering and so on. The dynamical behaviors of Eq. (1.1) have been extensively investigated (see [1–9] and the references therein).

In the case when $f \equiv 0$, Eq. (1.1) is a conservative system

$$x'' + g(x) = p(t). \tag{1.2}$$

It is well known that the time-map plays an important role in studying the existence of periodic solutions of Eq. (1.2) [10–14]. Assume that g satisfies the following condition,

(g) $\quad \lim_{|x|\to+\infty} \mathrm{sgn}(x)g(x) = +\infty.$

Set

$$\tau(c) = \sqrt{2}\left|\int_0^c \frac{ds}{\sqrt{G(c) - G(s)}}\right|.$$

☆ Research supported by China Postdoctoral Science Foundation and Beijing Natural Science Foundation (Existence and multiplicity of periodic solutions in nonlinear oscillations), No. 1112006.

* Tel.: +86 13426062458.
E-mail address: matiantian1215@163.com.

0362-546X/$ – see front matter
doi:10.1016/j.na.2012.03.004

The function τ is usually called the time-map related to the autonomous equation $x'' + g(x) = 0$. Let us define

$$\tau^+ = \limsup_{c\to+\infty} \tau(c); \qquad \tau_+ = \liminf_{c\to+\infty} \tau(c),$$

and

$$\tau^- = \limsup_{c\to-\infty} \tau(c); \qquad \tau_- = \liminf_{c\to-\infty} \tau(c).$$

By using the asymptotic property of time-map τ, Opial [10] obtained the following result.

Theorem A. *Assume that condition* (g) *holds and*

$$\tau_+ + \tau_- > T.$$

Then Eq. (1.2) *has at least one T-periodic solution.*

Theorem A was generalized by Fonda and Zanolin [12]. They proved the following theorem.

Theorem B. *Assume that condition* (g) *holds and either*

$$\tau^+ + \tau_- > T$$

or

$$\tau_+ + \tau^- > T$$

is satisfied. Then Eq. (1.2) *has at least one T-periodic solution.*

When $f \not\equiv 0$, the periodic problem of Eq. (1.1) is widely studied by using various methods such as topological degree, phase-plane analysis, continuation theorems and fixed point theorems. Assume that g satisfies the condition

(g′) $\quad \mathrm{sgn}(x)g(x) > \sup\{|f(t,0)| + |e(t)| : t \in R\}, \quad |x| \geq d,$

where d is a positive constant, and the primitive G ($G(x) = \int_0^x g(s)ds$) of g satisfies the condition

(G) $\quad \liminf_{x\to+\infty} \frac{2G(x)}{x^2} < \left(\frac{\pi}{T}\right)^2.$

Moreover, f satisfies the sublinear condition

(f) $\quad \lim_{|y|\to+\infty} \frac{f(t,y)}{y} = 0$

uniformly for $t \in [0, T]$. When conditions (g′), (G) and (f) hold, it was proved in [5] that Eq. (1.1) has at least one T-periodic solution.

In the present paper, we shall study the existence of periodic solutions of Eq. (1.1) by using the time-map τ defined above. We are able to prove the following result.

Theorem C. *Assume that conditions* (g), (f) *hold and either*

$$\tau^+ + \tau_- > T$$

or

$$\tau_+ + \tau^- > T$$

is satisfied. Then Eq. (1.1) *has at least one T-periodic solution.*

2. A continuation theorem

In order to deal with the existence of periodic solutions of Eq. (1.1), we shall introduce a continuation theorem for Rayleigh equations given in [5]. To this end, we consider the equivalent system of Eq. (1.1),

$$x' = y, \qquad y' = -(g(x) + f(t,y) - e(t)). \tag{2.1}$$

Now, we embed system (2.1) into a family of equations with one parameter $\lambda \in [0, 1]$,

$$x' = \lambda y, \qquad y' = -\lambda(g(x) + f(t,\lambda y) - e(t)). \tag{2.2}$$