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Renormings and the fixed point property in non-commutative L_1-spaces

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ABSTRACT

Let $\mathcal{M}$ be a finite von Neumann algebra. It is known that $L_1(\mathcal{M})$ and every non-reflexive subspace of $L_1(\mathcal{M})$ fail to have the fixed point property for non-expansive mappings (FPP). We prove a new fixed point theorem for this class of mappings in non-commutative $L_1(\mathcal{M})$ Banach spaces which lets us obtain a sufficient condition such that a closed subspace of $L_1(\mathcal{M})$ can be renormed to satisfy the FPP. As a consequence, we deduce that the predual of every atomic finite von Neumann algebra can be renormed with the FPP.

1. Introduction

Let X be a Banach space. It is said that X has the fixed point property (FPP) if every non-expansive mapping defined from a closed convex bounded subset into itself has a fixed point. The existence of fixed points for non-expansive mappings has been widely studied and it is well-known that the geometry of the Banach space plays a fundamental role in guaranteeing that there are fixed points for non-expansive mappings. For instance, if X is either uniformly convex or X is reflexive with normal structure, X satisfies the FPP, and classical non-reflexive Banach spaces, such as c_0 or ℓ_1, fail to have the FPP (for more background results the reader can consult [1] or [2] and the references therein). Many interesting problems are still open concerning the fixed point property and important results have been discovered in the last few years that have motivated new lines of research. For instance, Lin [3] has proved that there exists an equivalent norm $|||\cdot|||$ in the sequence space ℓ_1 such that $(\ell_1, |||\cdot|||)$ has the FPP. Lin's renorming answered, in a negative way, the long open question of whether FPP implies reflexivity. What is more, Lin's result connects renorming theory with fixed point theory in the following way: if a Banach space X fails the FPP with its original norm, can X be renormed to have this property? In this line, Domínguez Benavides [4] gave a positive answer in the case where the Banach space is reflexive. However, for non-reflexive Banach spaces we do not know a general solution. For instance, in [5] the authors proved that every renorming of ℓ_∞, $\ell_1(\Gamma)$ and $c_0(\Gamma)$, with Γ an uncountable set, contains an asymptotically isometric copy of either ℓ_1 or c_0 and, therefore, that these spaces cannot be renormed to have the FPP. In [6] the authors extend Lin's techniques in order to find new non-reflexive (and non-isomorphic to ℓ_1) Banach spaces failing the FPP that can be renormed to have this property and, using the convergence in measure topology, they give a sufficient condition such that a closed subspace of the measure space $L_1[0, 1]$ can also be renormed with the FPP. In this paper we focus on non-commutative L_1-spaces associated with finite von Neumann algebras.

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These spaces are the preduals of the von Neumann algebras and can be considered as an extension of the classical Lebesgue measure spaces (a particular example of a von Neumann algebra is $L_\infty[0, 1]$ and its predual is $L_1[0, 1]$). It is known that $L_1(\mathcal{M})$ and all its non-reflexive subspaces contain asymptotically isometric copies of ℓ_1 [7] and consequently they fail to have the FPP [5]. The main object of this paper is to obtain a family of norms equivalent to the usual norm in $L_1(\mathcal{M})$, which have a better behaviour with respect to the existence of fixed points for non-expansive mappings defined on a closed convex bounded subset of $L_1(\mathcal{M})$ and we give a sufficient condition (with a topological flavour) such that a non-reflexive subspace of $L_1(\mathcal{M})$ can be renormed with the FPP. As a consequence we will cover all the examples given in Section 5 of [6] for the particular case of the measure space $L_1(\mu)$ and we will deduce that if $\mathcal{M}$ is any atomic finite von Neumann algebra, there exist equivalent norms in $L_1(\mathcal{M})$ satisfying the fixed point property.

2. Non-commutative background

All the definitions and results of this section can be found in [8,9] and the Appendix of [10]. Two important references for non-commutative L_p-spaces and non-commutative integration are [11,12].

Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Some important locally convex topologies can be considered on $B(H)$: the uniform topology, given by the operator norm $\|x\|_\infty = \sup\{\|xh\| : h \in H, \|h\| \leq 1\}$; the strong operator topology, defined by the seminorms $x \to \|xh\|$ when h runs over H; and the weak operator topology, defined by the family of seminorms $x \to |(xh, g)|$ where $h, g \in H$.

Definition 1. A *von Neumann algebra* is a subalgebra $\mathcal{M}$ of $B(H)$ which is self-adjoint ($x \in \mathcal{M}$ implies $x^* \in \mathcal{M}$), contains I (the identity operator) and is closed in the weak operator topology (or equivalently, is closed in the strong operator topology).

The von Neumann algebras were introduced by Murray and von Neumann in their famous work [13] motivated by problems in infinite dimensional representation of groups and the mathematical foundations of quantum mechanics. In the case where H is a separable infinite dimensional Hilbert space and $(e_n)_n$ is an orthonormal basis in H, every $T \in B(H)$ has a matrix representation in the form

$$T = \big((Te_i, e_j)\big)_{i\geq 1; j\geq 1}.$$

Hence a von Neumann algebra is a unital sub*-algebra of $B(H)$ which is closed in the topology of the coordinate-to-coordinate convergence (which turns out to be the weak operator topology).

A first example of a von Neumann algebra is $B(H)$ itself. If H is a finite dimensional Hilbert space with $\dim(H) = n$, $B(H)$ coincides with $\mathcal{M}_n$, the set of all complex matrices with dimension $n \times n$.

Other simple examples of von Neumann algebras are ℓ_∞ and $L_\infty(\mu)$. In fact ℓ_∞ can be seen as the subalgebra of all operators in $B(\ell_2)$ that have a diagonal matrix. If (Ω, Σ, μ) is a measure space, every $f \in L_\infty(\mu)$ gives an operator on $L_2(\mu)$ given by

$$M_f : L_2(\mu) \to L_2(\mu); \qquad g \in L_2(\mu) \to M_f(g) = f \cdot g \in L_2(\mu)$$

and $L_\infty(\mu)$ is the von Neumann algebra acting in the Hilbert space $L_2(\mu)$ and given by the operators M_f with $f \in L_\infty(\mu)$. In fact, it is a classical result (see for instance Chapter 7 in [14]) that every commutative von Neumann algebra $\mathcal{M}$ can be isometrically identified with $L_\infty(\Omega, \mathcal{A}, \mu)$ for some abstract measure space $(\Omega, \mathcal{A}, \mu)$.

An important object in the study of von Neumann algebras is the collection of all orthogonal projections in $\mathcal{M}$, which is denoted by $\mathcal{P}(\mathcal{M})$. It is the analogue in non-commutative integration theory of the underlying σ-algebra in classical integration theory. If $p, q \in \mathcal{P}(\mathcal{M})$, then $p \leq q$ if and only if $\mathrm{Ran}(p) \subset \mathrm{Ran}(q)$. For $p, q \in \mathcal{P}(\mathcal{M})$ the supremum $p \bigvee q \in \mathcal{P}(\mathcal{M})$ and the infimum $p \bigwedge q \in \mathcal{P}(\mathcal{M})$ are given by the orthogonal projections onto $\overline{\mathrm{Ran}(p) + \mathrm{Ran}(q)}$ and $\mathrm{Ran}(p) \cap \mathrm{Ran}(q)$ respectively. Actually, $\mathcal{P}(\mathcal{M})$ is a complete lattice, that is, for each collection $\{p_\alpha\}$ in $\mathcal{P}(\mathcal{M})$, the supremum $\bigvee_\alpha p_\alpha$ and infimum $\bigwedge_\alpha p_\alpha$ exist and are given by the orthogonal projections onto $\overline{\mathrm{span}}_\alpha\{\mathrm{Ran}(p_\alpha)\}$ and $\bigcap_\alpha \mathrm{Ran}(p_\alpha)$ respectively. If $p \in \mathcal{P}(\mathcal{M})$, let $p^\perp = 1 - p \in \mathcal{P}(\mathcal{M})$. A projection $p \in \mathcal{P}(\mathcal{M})$ is called a minimal projection if there exists no $q \in \mathcal{P}(\mathcal{M})$ with $q \leq p$ and $0 \neq q \neq p$. If x is a self-adjoint element of $\mathcal{M}$ and A is a Borel subset of $\mathbb{R}$, we denote by e_A^x the spectral projection of x that corresponds to the subset A (see for instance [8, Chapter 2]).

Two projections e and f in a von Neumann algebras are said to be equivalent if there exists an element $u \in \mathcal{M}$ with $uu^\star = e$ and $u^\star u = f$. This fact is denoted by $e \sim f$. A projection e is said to be finite if $e \sim f \leq e$ implies $e = f$.

A von Neumann algebra $\mathcal{M}$ is said to be finite if the identity I is a finite projection, which is equivalent to saying that the hypotheses $T \in \mathcal{M}$ and $TT^\star = \mathrm{I}$ imply that $T^\star T = \mathrm{I}$ (see [14], Part III, Chapter 8, Theorem 1). It is clear that every commutative von Neumann algebra is finite.

Let $\mathcal{M}_+$ be the cone of all positive elements of $\mathcal{M}$, that is,

$$\mathcal{M}_+ = \{x \in \mathcal{M} : \langle xh|h\rangle \geq 0, \text{ for all } h \in H\}.$$

Definition 2. Let $\mathcal{M}$ be a von Neumann algebra. A trace on $\mathcal{M}$ is a map $\tau : \mathcal{M}_+ \to [0, \infty]$ satisfying

(1) $\tau(x + y) = \tau(x) + \tau(y)$, for all $x, y \in \mathcal{M}_+$.
(2) $\tau(\lambda x) = \lambda\tau(x)$; $x \in \mathcal{M}_+$, $\lambda \in [0, +\infty]$ (with the usual convention that $0 \cdot \infty = 0$).
(3) $\tau(xx^*) = \tau(x^*x)$ for all $x \in \mathcal{M}$.