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Asymptotics for a free boundary model in price formation

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ABSTRACT

We study the asymptotics for a large time of solutions to a one-dimensional parabolic evolution equation with non-standard measure-valued right hand side, that involves derivatives of the solution computed at a free boundary point. The problem is a particular case of a mean-field free boundary model proposed by Lasry–Lions on price formation and dynamic equilibria.

The main step in the proof is based on the fact that the free boundary disappears in the linearized problem, thus it can be treated as a perturbation through semigroup theory. This requires a delicate choice for the function spaces since higher regularity is needed near the free boundary. We show global existence for solutions with initial data in a small neighborhood of any equilibrium point, and exponential decay towards a stationary state. Moreover, the family of equilibria of the equation is stable, as follows from center manifold theory.

1. Introduction

We consider an idealized population of players consisting of two groups, namely one group of buyers of a certain good and one group of vendors of the same good. The two groups are described by two non-negative densities f_B, f_V depending on $(x, t) \in \mathbb{R} \times \mathbb{R}_+$. In the model, x denotes a possible value of the price and t the time.

At a certain time t, the vendors would like to sell the good, and the function $f_V(x, t)$ describes the density of the vendors who are willing to sell the good at price x. Meanwhile the buyers will try to get the good at a cheaper price. The transaction takes place when the two groups agree on the price: we denote by $p(t)$ the *agreement* price. The price $p(t)$ will be the highest price the buyers are willing to pay, and the lowest price the vendors agreed to sell the good. There exists a transaction cost, which is denoted by a positive constant a. When a buyer gets the good for the price $p(t)$, the actual cost of his trade is $p(t)+a$, as well as the profit for the seller is $p(t) - a$. As a consequence, the buyer that got the good for the price $p(t)$, will try in a later time to sell the good at least at the price $p(t) + a$ and the vendor that sold the good for $p(t)$ will try to get at a later time the same good for a price not higher than $p(t) - a$. Thus the parameter a introduces some friction in the system.

The price $p(t)$ results from a dynamical equilibrium between the two density functions. The randomness in the problem is measured by the diffusion coefficient of the two densities f_V and f_B, and is denoted by a parameter $\sigma > 0$.

The above situation can be described by the following system of free boundary evolution equations:

$$\begin{cases} \dfrac{\partial f_B}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 f_B}{\partial x^2} = \lambda(t)\delta_{x=p(t)-a} \quad \text{if } x \le p(t),\ t > 0, \\ f_B(x,t) > 0 \quad \text{if } x < p(t), \qquad f_B(x,t) = 0 \quad \text{if } x \ge p(t), \end{cases} \tag{1.1}$$

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together with

$$\begin{cases} \dfrac{\partial f_V}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 f_V}{\partial x^2} = \lambda(t)\delta_{x=p(t)+a} \quad \text{if } x > p(t),\ t > 0, \\ f_V(x,t) > 0 \quad \text{if } x > p(t), \qquad f_V(x,t) = 0 \quad \text{if } x \le p(t), \end{cases} \tag{1.2}$$

where

$$\lambda(t) = -\frac{\sigma^2}{2}\frac{\partial f_B}{\partial x}(p(t),t) = \frac{\sigma^2}{2}\frac{\partial f_V}{\partial x}(p(t),t). \tag{1.3}$$

The symbol δ denotes the Dirac delta at the indicated point. The multiplier $\lambda(t)$ represents the number of transactions at time t, so (1.3) means that the flux of buyers must be equal to the flux of vendors. The initial conditions

$$f_B(x,0) = f_B^I(x) \quad \text{and} \quad f_V(x,0) = f_V^I(x)$$

are such that, for some p_I in $\mathbb{R}$,

$$f_B^I(x) > 0 \quad \text{if } x < p_I, \qquad f_B^I(x) = 0 \quad \text{if } x \ge p_I$$
$$f_V^I(x) > 0 \quad \text{if } x > p_I, \qquad f_B^I(x) = 0 \quad \text{if } x \le p_I.$$

The equations satisfy the property of conservation of mass. Indeed, both

$$\int_{-\infty}^{p(t)} f_V \,\mathrm{d}x \quad \text{and} \quad \int_{p(t)}^{+\infty} f_B \,\mathrm{d}x$$

remain constant for all $t \ge 0$.

Eqs. (1.1)–(1.3) describe a mean-field model for the dynamical formation of the price of some good that has been recently introduced in [1].

An important question we are going to address here concerns the long time behavior of the system: will the good reach a stable price ($p(t) \to \text{const.}$ as $t \to \infty$?) or will the price keep oscillating in time and never reach a stable value?

We remark here that in a bounded interval with symmetric initial data, the solution remains symmetric for all times and the asymptotics were proved by the authors in their previous work [2]. However, the general case contains a new ingredient: a free boundary (see [3], for instance, for some background and examples on these type of problems).

In this work we address the problem (1.1)–(1.3) in a bounded interval $[-A, B]$, $A, B > 0$, for $a < \min\{A/2, B/2\}$, with zero Neumann boundary conditions. The aim is to show that if we start with an initial condition that is near a general equilibrium point in some suitable function space, then there exists a unique solution of (1.1)–(1.3) that decays exponentially fast in time to a unique stationary state. In addition, in can be shown that the problem presents a two-dimensional family of equilibria, and that this family is stable.

Although there is a well developed theory of semigroups and invariant manifolds for the study of evolution equations (see for instance [4–7]), the main novelty here is the fact that dynamical system arguments can be used for a problem that presents a free boundary. This is possible since we succeeded to treat the free boundary as a perturbation of the linearized problem. In fact, as we will see in the following sections, the free boundary disappears in the linearization and appears again in the nonlinear part of the problem as a term of lower order.

This allows us to study the linearized operator with the classical semigroup theory and to get time estimates for the linear equation. Even though the linearized operator is a non-standard one, we can explicitly compute its eigenvalues and corresponding eigenfunctions. Unfortunately the eigenfunctions do not build an orthogonal basis with respect to the standard product in L^2. This complicates the choice of functional spaces.

Indeed, the choice of function spaces is a delicate step in the proof. They need to be big enough to allow delta functions in the equation, but on the other hand, higher regularity is needed near the free boundary. In order to give a more explicit characterization of those spaces, interpolation theory and pseudo-differential operators are needed.

For simplicity of the notation, we rewrite the problem (1.1)–(1.3) as the single equation

$$\begin{cases} \dfrac{\partial f}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 f}{\partial x^2} = \lambda(t)\left[\delta_{x=p(t)-a} - \delta_{x=p(t)+a}\right] \quad \text{in } [-A,B] \times \mathbb{R}_+, \\ f(x,0) = f_I(x) \quad \text{in } [-A,B], \\ f_x(-A,0) = f_x(B,0) = 0, \end{cases} \tag{1.4}$$

where

$$\lambda(t) := -\frac{\sigma^2}{2}\frac{\partial f}{\partial x}(p(t),t),$$

and

$$f := f_B - f_V, \qquad f_I = f_B^I - f_V^I, \quad \text{and} \quad p(0) = p_I.$$