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Fractional variational problems depending on indefinite integrals[☆]

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ABSTRACT

We obtain necessary optimality conditions for variational problems with a Lagrangian depending on a Caputo fractional derivative, a fractional and an indefinite integral. Main results give fractional Euler–Lagrange type equations and natural boundary conditions, which provide a generalization of the previous results found in the literature. Isoperimetric problems, problems with holonomic constraints and depending on higher-order Caputo derivatives, as well as fractional Lagrange problems, are considered.

1. Introduction

In the 18th century, Euler considered the problem of optimizing functionals depending not only on some unknown function y and some derivative of y, but also on an antiderivative of y (see [1]). Similar problems have been recently investigated in [2], where Lagrangians containing higher-order derivatives and optimal control problems are considered. More generally, it has been shown that the results of [2] hold on an arbitrary time scale [3]. Here we study such problems within the framework of fractional calculus.

Roughly speaking, a fractional calculus defines integrals and derivatives of non-integer order. Let $\alpha > 0$ be a real number and $n \in \mathbb{N}$ be such that $n - 1 < \alpha < n$. Here we follow [4] and [5,6]. Let $f : [a, b] \to \mathbb{R}$ be piecewise continuous on (a, b) and integrable on $[a, b]$. The left and right Riemann–Liouville fractional integrals of f of order α are defined respectively by

$$ {}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t \quad \text{and} \quad {}_xI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t. $$

Here Γ is the well-known Gamma function. Then the left ${}_aD_x^\alpha$ and right ${}_xD_b^\alpha$ Riemann–Liouville fractional derivatives of f of order α are defined (if they exist) as

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_a^x (x-t)^{n-\alpha-1} f(t)\mathrm{d}t \tag{1} $$

and

$$ {}_xD_b^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^b (t-x)^{n-\alpha-1} f(t)\mathrm{d}t. \tag{2} $$

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The fractional derivatives (1) and (2) have one disadvantage when modeling real world phenomena: the fractional derivative of a constant is not zero. To eliminate this problem, one often considers fractional derivatives in the sense of Caputo. Let f belong to the space $AC^n([a,b];\mathbb{R})$ of absolutely continuous functions. The left and right Caputo fractional derivatives of f of order α are defined respectively by

$$ {}_a^C D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\mathrm{d}t $$

and

$$ {}_x^C D_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (-1)^n (t-x)^{n-\alpha-1} f^{(n)}(t)\mathrm{d}t. $$

These fractional integrals and derivatives define a rich calculus. For details see the books [5–7]. Here we just recall a useful property for our purposes: integration by parts. For fractional integrals,

$$ \int_a^b g(x) \cdot {}_a I_x^\alpha f(x)\mathrm{d}x = \int_a^b f(x) \cdot {}_x I_b^\alpha g(x)\mathrm{d}x $$

(see, e.g., [5, Lemma 2.7]), and for Caputo fractional derivatives

$$ \int_a^b g(x) \cdot {}_a^C D_x^\alpha f(x)\mathrm{d}x = \int_a^b f(x) \cdot {}_x D_b^\alpha g(x)\mathrm{d}x + \sum_{j=0}^{n-1} \left[{}_x D_b^{\alpha+j-n} g(x) \cdot f^{(n-1-j)}(x) \right]_a^b $$

(see, e.g., [8, Eq. (16)]). In particular, for $\alpha \in (0,1)$ one has

$$ \int_a^b g(x) \cdot {}_a^C D_x^\alpha f(x)\mathrm{d}x = \int_a^b f(x) \cdot {}_x D_b^\alpha g(x)\mathrm{d}x + \left[{}_x I_b^{1-\alpha} g(x) \cdot f(x) \right]_a^b. \tag{3} $$

When $\alpha \to 1$, ${}_a^C D_x^\alpha = \frac{\mathrm{d}}{\mathrm{d}x}$, ${}_x D_b^\alpha = -\frac{\mathrm{d}}{\mathrm{d}x}$, ${}_x I_b^{1-\alpha}$ is the identity operator, and (3) gives the classical formula of integration by parts.

The fractional calculus of variations concerns finding extremizers for variational functionals depending on fractional derivatives instead of integer ones. The theory started in 1996 with the work of Riewe, in order to better describe nonconservative systems in mechanics [9,10]. The subject is now under strong development due to its many applications in physics and engineering, providing more accurate models of physical phenomena (see, e.g., [11–20]). With respect to results on fractional variational calculus via Caputo operators, we refer the reader to [21–27] and references therein.

Our main contribution is an extension of the results presented in [2,21] by considering Lagrangians containing an antiderivative, that in turn depend on the unknown function, a fractional integral, and a Caputo fractional derivative (Section 2). Transversality conditions are studied in Section 3, where the variational functional J depends also on the terminal time T, $J(y,T)$, and where we obtain conditions for a pair (y,T) to be an optimal solution to the problem. In Section 4 we consider isoperimetric problems with integral constraints of the same type as the cost functionals considered in Section 2. Fractional problems with holonomic constraints are considered in Section 5. The situation when the Lagrangian depends on higher-order Caputo derivatives, i.e., it depends on ${}_a^C D_x^{\alpha_k} y(x)$ for $\alpha_k \in (k-1,k)$, $k \in \{1,\ldots,n\}$, is studied in Section 6, while Section 7 considers fractional Lagrange problems and the Hamiltonian approach. In Section 8 we obtain sufficient conditions of optimization under suitable convexity assumptions on the Lagrangian. We end with Section 9, discussing a numerical scheme for solving the proposed fractional variational problems. The idea is to approximate fractional problems by classical ones. Numerical results for two illustrative examples are described in detail.

2. The fundamental problem

Let $\alpha \in (0,1)$ and $\beta > 0$. The problem that we address is stated in the following way. Minimize the cost functional

$$ J(y) = \int_a^b L(x, y(x), {}_a^C D_x^\alpha y(x), {}_a I_x^\beta y(x), z(x))\mathrm{d}x, \tag{4} $$

where the variable z is defined by

$$ z(x) = \int_a^x l(t, y(t), {}_a^C D_t^\alpha y(t), {}_a I_t^\beta y(t))\mathrm{d}t, $$

subject to the boundary conditions

$$ y(a) = y_a \quad \text{and} \quad y(b) = y_b. \tag{5} $$

We assume that the functions $(x,y,v,w,z) \to L(x,y,v,w,z)$ and $(x,y,v,w) \to l(x,y,v,w)$ are of class C^1, and the trajectories $y : [a,b] \to \mathbb{R}$ are absolute continuous functions, $y \in AC([a,b];\mathbb{R})$, such that ${}_a^C D_x^\alpha y(x)$ and ${}_a I_x^\beta y(x)$ exist and are continuous on $[a,b]$. We denote such class of functions by $\mathcal{F}([a,b];\mathbb{R})$. Also, to simplify, by $[\cdot]$ and $\{\cdot\}$ we denote the operators

$$ [y](x) = (x, y(x), {}_a^C D_x^\alpha y(x), {}_a I_x^\beta y(x), z(x)) \quad \text{and} \quad \{y\}(x) = (x, y(x), {}_a^C D_x^\alpha y(x), {}_a I_x^\beta y(x)). $$