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Error bounds for vector-valued functions: Necessary and sufficient conditions

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ABSTRACT

In this paper, we attempt to extend the definition and existing local error bound criteria to vector-valued functions, or more generally, to functions taking values in a normed linear space. Some new derivative-like objects (slopes and subdifferentials) are introduced and a general classification scheme of error bound criteria is presented.

1. Introduction

In variational analysis, the term "*error bounds*" usually refers to the following property. Given an (extended) real-valued function f on a set X, consider its lower level set

$$S(f) := \{x \in X \,|\, f(x) \le 0\} \quad - \tag{1.1}$$

the set of all solutions of the inequality $f(x) \le 0$. If a point x is not a solution, that is, $f(x) > 0$, then it can be important to have an estimate of its distance from the set (1.1) (assuming that X is a metric space) in terms of the value $f(x)$. If a linear estimate is possible, that is, there exists a constant $\gamma > 0$ such that

$$d(x, S(f)) \le \gamma f^+(x) \tag{1.2}$$

for all $x \in X$, then f possesses the (linear) error bound property or the error bound property holds for f. Here the denotation $f^+(x) = \max(f(x), 0)$ is used. Hence, (1.2) is satisfied automatically for all $x \in S(f)$.

If $\bar{x} \in S(f)$ (usually it is assumed that $f(\bar{x}) = 0$) and (1.2) is required to hold (with some $\gamma > 0$) for all x near $\bar{x}$, then we have the definition of the local (near $\bar{x}$) error bound property.

Error bounds play a key role in variational analysis. They are of great importance for optimality conditions, subdifferential calculus, stability and sensitivity issues, convergence of numerical methods, etc. For the summary of the theory of error bounds and its various applications to sensitivity analysis, convergence analysis of algorithms, and penalty function methods

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in mathematical programming, the reader is referred to the survey papers by Azé [1], Lewis and Pang [2], Pang [3], as well as the book by Auslender and Teboulle [4].

Numerous characterizations of the error bound property have been established in terms of various derivative-like objects either in the primal space (directional derivatives, slopes, etc.) or in the dual space (subdifferentials, normal cones) [3,5–27].

In the present paper, we attempt to extend (1.2) as well as local error bound criteria to vector-valued functions f, defined on a metric space X and taking values in a normed linear space Y. The presentation, terminology and notation follow that of [25]. Some new derivative-like objects (slopes and subdifferentials), which can be of independent interest, are introduced and a general classification scheme of error bound criteria is presented. It is illustrated in Figs. 3–8.

The plan of the paper is as follows. In Section 2, we introduce an abstract ordering operation and define an extension of (1.1) and a nonnegative real-valued function f_y^+ whose role is to replace f^+ in (1.2) in the case f takes values in a normed linear space. Note that, unlike the scalar case, an additional parameter y is required now. The function f_y^+ can be viewed as a *scalarizing* function for the vector minimization problem defined by f. In Sections 3 and 4, various kinds of slopes, directional derivatives and subdifferentials for vector-valued function f are defined via the scalarizing function f_y^+. In Section 3, we discuss primal space error bound criteria in terms of slopes. Section 4 is devoted to the criteria in terms of directional derivatives and subdifferentials. In the final Section 5, three special cases are considered: error bounds when either the image or the preimage is finite dimensional and in the convex case.

Our basic notation is standard, see [28,29]. Depending on the context, X is either a metric or a normed space. Y is always a normed space. If X is a normed space, then its topological dual is denoted as X^* while $\langle\cdot,\cdot\rangle$ denotes the bilinear form defining the pairing between the two spaces. The closed unit balls in a normed space and its dual are denoted by $\mathbb{B}$ and $\mathbb{B}^*$ respectively. $B_\delta(x)$ denotes the closed ball with radius δ and center x. If A is a set in metric or normed space, then cl A, int A, and bd A denote its closure, interior, and boundary respectively; $d(x, A) = \inf_{a\in A}\|x-a\|$ is the point-to-set distance. We also use the denotation $\alpha^+ = \max(\alpha, 0)$.

2. Minimality and error bounds

In this section, an ordering operation in a normed linear space is discussed and the error bound property is defined.

2.1. Minimality

Let Y be a normed linear space. Suppose that for each $y \in Y$, a subset $V_y \subset Y$ is given with the property $y \in V_y$. We are going to consider an abstract "order" operation in Y defined by the collection of sets $\{V_y\}$: we say that v is dominated by y in Y if $v \in V_y$.

Of course, this operation does not possess in general typical order properties. It can get more natural, for example, if a closed cone $C \subset Y$ is given and V_y is defined as one of the following (for (2.3) C must be assumed pointed):

$$\{v \in Y \,|\, y - v \in C\}, \tag{2.1}$$

$$\{v \in Y \,|\, v - y \notin C\} \cup \{y\}, \tag{2.2}$$

$$\{v \in Y \,|\, v - y \notin \operatorname{int} C\}. \tag{2.3}$$

Now, let X be a metric space and $f : X \to Y$. Denote

$$S_y(f) := \{u \in X \,|\, f(u) \in V_y\} \quad -$$

the y-sublevel set of f (with respect to the order defined by the collection of sets $\{V_y\}$). We will also use the following nonnegative real-valued function

$$f_y^+(u) := d(f(u), V_y), \quad u \in X. \tag{2.4}$$

If V_y is closed, then $f_y^+(u) = 0$ if and only if $u \in S_y(f)$.

We say that x is a local V_y-*minimal* point of f if

$$f_y^+(x) \le f_y^+(u) \quad \text{for all } u \text{ near } x. \tag{2.5}$$

The definition depends on the choice of y. Condition (2.5) is obviously satisfied for any $x \in S_y(f)$.

If V_y is given by (2.1), i.e., $V_y = y - C$, then $f_y^+(u) = d(y - f(u), C)$ and the function f_y^+ is nondecreasing in the sense that for any $x_1, x_2 \in X$ such that $f(x_2) - f(x_1) \in C$ it holds $f_y^+(x_1) \le f_y^+(x_2)$. Indeed,

$$\begin{aligned} f_y^+(x_1) &= d(y - f(x_1), C) \le d(y - f(x_2), C) + d(f(x_2) - f(x_1), C) \\ &= d(y - f(x_2), C) = f_y^+(x_2). \end{aligned}$$

By this property, if C is a pointed cone and x is a *strict local* V_y-*minimal* point of f, i.e.,

$$f_y^+(x) < f_y^+(u) \quad \text{for all } u \text{ near } x,\ u \ne x,$$