Nonlinear Analysis 75 (2012) 1141–1146

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

An upper estimate for the Clarke subdifferential of an infimal value function proved via the Mordukhovich subdifferential

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ARTICLE INFO

Communicated by Enzo Mitidieri

MSC:
49J52
90C26
90C56

Keywords:
Infimal value function
Clarke subdifferential
Mordukhovich subdifferential

ABSTRACT

The aim of this note is to give an alternative proof for a recent result due to Dorsch et al., which provides an upper estimate for the Clarke subdifferential of an infimal value function. We show the validity of this result under a weaker condition than the one assumed in the aforementioned paper, while the use of the Mordukhovich subdifferential, as an intermediate step, will considerably shorten its proof.

1. Motivation and preliminary results

The setting that we work within in this article, which is in fact the one considered in [1], is the following. Let $g_0, \ldots, g_s : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be real-valued and continuously differentiable functions and $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ be defined as

$$\varphi(x) := \inf_{y \in \mathbb{R}^m} \max_{0 \le k \le s} g_k(x, y).$$

The topological structure of the upper-level set

$$M^{\max} := \{x \in \mathbb{R}^n : \varphi(x) \ge 0\}$$

of the *infimal value function* φ is of particular interest in the study of generalized semi-infinite optimization problems (cf. [2]).

Further, let us have

$$\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \sigma(x, y) := \max_{0 \le k \le s} g_k(x, y),$$

the set-valued operator

$$M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m, \qquad M(x) := \{y \in \mathbb{R}^m : \sigma(x, y) = \varphi(x)\}$$

and for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ the following set of indices:

$$K(x, y) := \{k \in \{0, \ldots, s\} : g_k(x, y) = \sigma(x, y)\}.$$

For $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ we consider (cf. [2]) the convex and compact set

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0362-546X/$ – see front matter
doi:10.1016/j.na.2011.08.009

$$V(x,y) := \left\{ \sum_{k\in K(x,y)} \mu_k D_x g_k(x,y) \left| \begin{array}{l} \sum_{k\in K(x,y)} \mu_k D_y g_k(x,y) = 0, \\ \sum_{k\in K(x,y)} \mu_k = 1, \\ \mu_k \geq 0 \quad \forall k \in K(x,y) \end{array} \right. \right\}.$$

Here, for a function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $D_x g$ and $D_y g$ denote the *gradients* of g with respect to the variables x, respectively, y. Further, let us have

$$V : \mathbb{R}^n \rightrightarrows \mathbb{R}^m, \qquad V(x) := \bigcup_{y\in M(x)} V(x,y).$$

The following condition was introduced in [1].

Compactness Condition CC. One says that Compactness Condition CC is fulfilled if for all sequences $(x_k, y_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ with

$$\begin{cases} \bullet \quad x_k \to x \in \mathbb{R}^n \ (k \to \infty) \\ \bullet \quad \text{either } \sigma(x_k, y_k) \to a \ (k \to \infty) \quad \text{and} \quad a \leq \varphi(x) \\ \quad\ \ \text{or} \quad \sigma(x_k, y_k) \to -\infty \ (k \to \infty) \end{cases}$$

the sequence $(y_k)_{k\in\mathbb{N}}$ contains a convergent subsequence.

One of the main results of [1] is represented by the following upper estimate for the *Clarke subdifferential* of the function φ.

Theorem 1. *Let Compactness Condition CC be fulfilled and let $\bar{x} \in \mathbb{R}^n$. Then it holds that*

$$\partial^C \varphi(\bar{x}) \subseteq \operatorname{conv} V(\bar{x}). \tag{1}$$

In the above result, $\partial^C \varphi(\bar{x})$ denotes the Clarke subdifferential of φ at $\bar{x}$, while $\operatorname{conv} V(\bar{x})$ is the convex hull of the set $V(\bar{x})$.

The proof given in [1] for this result is quite involved and makes use of some characterizations of the Clarke subdifferential from [3]. We will give in this note an alternative proof for the above inclusion under a weaker assumption than Compactness Condition CC, by using as an intermediate tool the *Mordukhovich subdifferential*. This proof will allow us to point out what are the difficulties one has to face when trying to discuss the situation when the inclusion in Theorem 1 becomes an equality.

The condition which will turn out to be sufficient for (1) was given in [1], too, and has the following formulation.

Condition C^*. *One says that Condition C^* is fulfilled if*

(C1) *for all $x \in \mathbb{R}^n$ and sequences $(y_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^m$ with $\sigma(x, y_k) \to \varphi(x)(k \to \infty)$ there exists a convergent subsequence of $(y_k)_{k\in\mathbb{N}}$ and*

(C2) *the mapping $x \rightrightarrows M(x)$ is locally bounded, i.e., for all $\bar{x} \in \mathbb{R}^n$ there exists an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ such that $\bigcup_{x\in U_{\bar{x}}} M(x)$ is bounded.*

According to [1, Lemma 2.1], Condition C^* guarantees the following local description of φ: for every $\bar{x} \in \mathbb{R}^n$ there exists an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ and a compact set $W \subseteq \mathbb{R}^m$ such that

$$\varphi(x) = \min_{y\in W} \sigma(x,y) \quad \forall x \in U_{\bar{x}}. \tag{2}$$

As proved in [1], Condition C^* is implied by Compactness Condition CC and the two conditions are not equivalent. However, according to [1], Condition C^* is not stable with respect to C^0-perturbations of the functions involved, which is not the case for Compactness Condition CC. Nevertheless, for guaranteeing (1), one does not necessarily have to assume that the latter is fulfilled, as we will prove in the next section. To this end, we need several notions and results, which we introduce in the following.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a given function with a nonempty *effective domain* $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. The *epigraph* of f is the set $\operatorname{epi} f := \{(x,r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$. We say that f is *lower semicontinuous around* $\bar{x} \in \operatorname{dom} f$ if there exists an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ such that f is lower semicontinuous at x for all $x \in U_{\bar{x}}$. We say that f is *locally Lipschitzian around* $\bar{x} \in \operatorname{dom} f$ if there exists an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ and a real number $L > 0$ such that $|f(x) - f(y)| \leq L\|x - y\|$ for all $x, y \in U_{\bar{x}}$.

For $\varepsilon \geq 0$ the *Fréchet ε-subdifferential* (or the *analytic ε-subdifferential*) of f at $\bar{x} \in \operatorname{dom} f$ is defined by

$$\partial^F_\varepsilon f(\bar{x}) := \left\{ x^* \in X^* : \liminf_{\|h\|\to 0} \frac{f(\bar{x}+h) - f(\bar{x}) - \langle x^*, h\rangle}{\|h\|} \geq -\varepsilon \right\},$$

while for $\bar{x} \notin \operatorname{dom} f$ we set $\partial^F_\varepsilon f(\bar{x}) := \emptyset$. Further, $\partial^F f(\bar{x}) := \partial^F_0 f(\bar{x})$ denotes the classical *Fréchet subdifferential* of f at $\bar{x}$.