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A study of tilt-stable optimality and sufficient conditions

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ABSTRACT

Recent results by Eberhard et al. (2006) [4] and Eberhard and Wenczel (2009) [3] on the interaction of single- and double-envelope operations of nonsmooth functions and their interaction with second-order derivations have been used to study tilt-stability of local minima. This continues the study begun by Poliquin and Rockafellar (1998) [1] but now, armed with new tools we are able to make some new observations. We observe that tilt-stability entails a local density within the graph of the proximal subderivative of strict local minima order two of the tilted function. Indeed, it also entails the strict local minimality (order two) of the tilt-stable local minimum itself. For prox-regular, subdifferentially continuous functions this density property characterises tilt stability.

1. Introduction

The work by Poliquin and Rockafellar [1,2] has shown that for a large class of nonsmooth functions a second-order necessary condition based on the coderivative to the subdifferential characterises a tilt-stable local minimum. In this paper, we continue the work of [1,3] and explore the use of a regularisation operation to derive sufficiency conditions via the approximation of nonsmooth functions derived from smoother functions associated with lower-envelope operations. This was essentially the approach used in [1] but more recent work in [4,5,3] has provided new tools which necessitate the revisiting of the study of sufficient optimality for tilt-stable minima.

If we lift a negative-quadratic so that it touches the epigraph of $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $\bar{x} \in \operatorname{dom} f$, we obtain the proximal-subdifferential condition

$$f(x) \geq f(\bar{x}) + \langle p, x - \bar{x}\rangle - \frac{r}{2}\|x - \bar{x}\|^2. \tag{1}$$

Denote by $\partial_p f(\bar{x})$ the proximal subdifferential, which consists of all p satisfying (1) in some neighbourhood of $\bar{x}$. If f is minorised by a quadratic and $\partial_p f(\bar{x}) \neq \emptyset$ then for r sufficiently large and positive we have (1) satisfied globally (see [6]). Denote by $r(f, \bar{x}, p)$ the infimum of all such numbers such that (1) holds for all x, and place $\bar{r}(f, \bar{x}, p) = \max\{0, r(f, \bar{x}, p)\}$.

In this paper, we almost exclusively concentrate on the case when optimality conditions are based on the proximal subdifferential $\partial_p f$. This is largely due to the fact that the Moreau envelopes interact in favourable ways with the proximal subdifferential.

The paper [1], and the earlier work [7], focus on optimality conditions based on the (basic) subdifferential of nonsmooth analysis, denoted by ∂f, as the primary object of the study. We note that the class of prox-regular, subdifferentially continuous functions as studied in [1] have the property that locally $\partial_p f = \partial f$ and hence the study entered into here is still comparable with that engaged in [1]. In this paper, we are concerned with the nature of additional second order sufficiency conditions that are required to obtain certain types of strict local minima.

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We say $\bar{x} \in C$ is a *strict local minimiser of order two* for f, *relative* to $B_\delta(\bar{x})$ (for a given $\delta > 0$), if there exist $\beta > 0$ such that

$$f(x) \geq f(\bar{x}) + \frac{\beta}{2}\|x - \bar{x}\|^2$$

for all $x \in B_\delta(\bar{x})$. In [3], a detailed study is made of sufficiency conditions that ensure the existence of strict local minima of order 2. In [1], the more structured notion of a tilt stable local minimum is studied and this study was extended to parametrised problems in [8]. In [9] a study is made of the role of very general compatible parametrisations and an associated stability theorem is proved. A point $\bar{x}$ is said to be a *tilt-stable local minimum* of the function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ if $f(\bar{x})$ is finite and there exists $\delta > 0$ such that the mapping

$$M : v \mapsto \operatorname*{argmin}_{|x-\bar{x}|\leq\delta} \{f(x) - f(\bar{x}) - \langle v, x - \bar{x}\rangle\}$$

is single-valued and Lipschitz-continuous on some neighbourhood of $v = 0$ with $M(0) = \bar{x}$.

For a multi-function $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ we denote its graph by $\operatorname{Graph} F := \{(x, y) \mid y \in F(x)\}$ and the *indicator function* $\delta_{\operatorname{Graph} F}(x, y)$ to be zero if $(x, y) \in \operatorname{Graph} F$ and $+\infty$, otherwise. The *Mordukhovich coderivative* is defined as

$$D^*F(x, y)(w) := \{p \in \mathbb{R}^n \mid (p, -w) \in \partial\delta_{\operatorname{Graph} F}(x, y) := N_{\operatorname{Graph} F}(x, y)\}.$$

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and assume the first-order condition $0 \in \partial_p f(\bar{x})$ holds. In [1], the second order sufficiency condition

$$\forall \|h\| = 1,\ p \in D^*\left(\partial_p f\right)(\bar{x}, 0)(h) \quad \text{we have } \langle p, h\rangle \geq \beta > 0 \tag{2}$$

is studied and shown to imply a tilt-stable local minimum when f is both prox-regular and subdifferentially continuous. We are interested in obtaining sufficient conditions that in the first case are very general and presume little about the function f, and then specialise these to the class of prox-regular, subdifferentially continuous functions. This allows us to embark on a study of the nature of tilt stable minima. Tilt-stable and strict-local-order-two minima have been studied separately in the literature but to our knowledge their relationship has not been addressed to date. The main contribution of this paper is to shed light on the relationships that exist between these two concepts.

This study necessitates the introduction of certain modified notions of coderivatives. The *λ-proximally stable graphical derivative* of $\partial_p f$ at $\bar{x}$ for $p \in \partial_p f(\bar{x})$ is given by:

$$D_\lambda(\partial_p f)(\bar{x}, p)(u) = \left\{ z \mid \exists z_n \to z, z_n \in \frac{1}{t_n}\left(\partial_p f\ (x + t_n u_n) - p\right) \text{ for some } (t_n, u_n) \right.$$
$$\left. \to (0_+, u) \text{ with } \lambda < \frac{1}{\bar{r}(f, \bar{x} + t_n u_n, p + t_n z_n)} \text{ for all } n \right\}. \tag{3}$$

The *λ-proximally stable tangent and normal cones* are given by:

$$T^\lambda_{\operatorname{Graph} \partial_p f}(\bar{x}, p) = \{(u, z) \mid z \in D_\lambda(\partial_p f)(\bar{x}, p)(u)\} = \operatorname{Graph} D_\lambda(\partial_p f)(\bar{x}, p)(\cdot) \quad \text{and}$$
$$\hat{N}^\lambda_{\operatorname{Graph} \partial_p f}(\bar{x}, p) = \left(T^\lambda_{\operatorname{Graph} \partial_p f}(\bar{x}, p)\right)^\circ, \tag{4}$$

where $\circ$ denotes the negative polar cone operation (see page 215–216 of [2]). Then we may define the *λ-proximally stable coderivative* by $\hat{D}^*_\lambda(\partial_p f)(\bar{x}, p)(u) := \left\{z \mid (z, -u) \in \hat{N}^\lambda_{\operatorname{Graph} \partial_p f}(\bar{x}, p)\right\}$. We refer the reader to Chapter 5 of [2] for relevant notions of graphical and total convergence used in the next definition.

Definition 1. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $(x, y) \in \operatorname{Graph} \partial_p f$ with $w \in \mathbb{R}^n$. Denote by $D^*_\lambda(\partial_p f)(x, y)(u) := \{z \mid (z, -w) \in \hat{N}^\lambda_{\operatorname{Graph} \partial_p f}(x, y)\}$ and define the *λ-filtered coderivative*

$$D^*_0(\partial_p f)(x, y)(w) := \bigcap_{\lambda > 0} g\text{-}\limsup_{(x', y')(\in \operatorname{Graph} \partial_p f) \to (x, y)} \hat{D}^*_\lambda\left(\partial_p f\right)(x', y')(w) \tag{5}$$

where we have taken the *graphical limit supremum* (see Definition 5.32 of [2]). Also, define

$$D^*_{\lambda,\infty}\left(\partial_p f\right)(x, y)(w) := \limsup^\infty_{\substack{(x', y')(\in \operatorname{Graph} \partial_p f) \to (x, y) \\ w' \to w}} \hat{D}^*_\lambda\left(\partial_p f\right)(x', y')(w') \quad \text{and}$$
$$D^*_\infty\left(\partial_p f\right)(x, y)(w) := \limsup^\infty_{w' \to w, \lambda \downarrow 0} \hat{D}^*_\lambda\left(\partial_p f\right)(x, y)(w) \cup \bigcap_{\lambda > 0} D^*_{\lambda,\infty}\left(\partial_p f\right)(x, y)(w). \tag{6}$$

As $\hat{D}^*_\lambda\left(\partial_p f\right)(x', y')(w) \supseteq \hat{D}^*\left(\partial_p f\right)(x', y')(w)$ we have $D^*_0\left(\partial_p f\right)(x, y)(w) \supseteq D^*\left(\partial_p f\right)(x, y)(w)$ for all w and when f is convex $\bar{r}(f, \bar{x} + t_n u_n, p + t_n z_n) = 0$ and so $D^*_\lambda\left(\partial_p f\right)(x, y) \equiv D^*\left(\partial_p f\right)(x, y)$ for all $\lambda > 0$.