Nonlinear Analysis 75 (2012) 1305–1323

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Optimality conditions for various efficient solutions involving coderivatives: From set-valued optimization problems to set-valued equilibrium problems

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ARTICLE INFO

Communicated by: Ravi Agarwal

Dedicated to the memory of my Teacher Professor I.A. Bakhtin.

MSC:
49J53
49K27
90C29
90C33

Keywords:
Set-valued optimization problem
Set-valued equilibrium problem
Vector equilibrium problem
Vector variational inequality
Normal cone
Coderivative
Necessary optimality conditions
Sufficient optimality conditions
Pareto efficient solutions
Weakly efficient solutions
Quasi weakly efficient solutions
Strongly efficient solutions
Properly efficient solutions
Relative efficient solutions

ABSTRACT

We present a new approach to the study of a set-valued equilibrium problem (for short, SEP) through the study of a set-valued optimization problem with a geometric constraint (for short, SOP) based on an equivalence between solutions of these problems. As illustrations, we adapt to SEP enhanced notions of relative Pareto efficient solutions introduced in set optimization by Bao and Mordukhovich and derive from known or new optimality conditions for various efficient solutions of SOP similar results for solutions of SEP as well as for solutions of a vector equilibrium problem and a vector variational inequality.

We also introduce the concept of quasi weakly efficient solutions for the above problems and divide all efficient solutions under consideration into the *Pareto-type group* containing Pareto efficient, primary relative efficient, intrinsic relative efficient, quasi relative efficient solutions and the *weak Pareto-type group* containing quasi weakly efficient, weakly efficient, strongly efficient, positive properly efficient, Henig global properly efficient, Henig properly efficient, super efficient and Benson properly efficient solutions. The necessary conditions for Pareto-type efficient solutions and necessary/sufficient conditions for weak Pareto-type efficient solutions formulated here are expressed in terms of the Ioffe approximate coderivative and normal cone in the Banach space setting and in terms of the Mordukhovich coderivative and normal cone in the Asplund space setting.

1. Introduction

Inspired by the pioneering work [1] by Giannessi on vector variational inequalities (for short, VVI), a scalar equilibrium problem has been extended to a vector equilibrium problem (for short, VEP) in [2] and to a set-valued equilibrium problem (for short, SEP) in [3]. SEP and its special cases, VEP and VVI, have applications in different areas of optimization, optimal control, operations research and economics. Numerous works have been devoted to the existence, scalarization, optimality conditions or connectedness of various efficient solutions of these problems; see, for instance, [4–24]. Note that most results are concerned with VEP and VVI and, to our knowledge, there are few papers which deal with optimality conditions for efficient solutions of SEP.

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doi:10.1016/j.na.2011.03.015

Since SEP is closely related to a set-valued optimization problem with a geometric constraint (for short, SOP), it is natural to borrow concepts, techniques and results on optimality conditions for the latter to try to come up with similar concepts and results for the former. Inspired by this idea and based on an equivalence between solutions of these problems, we present in this paper a new approach to the study of SEP through the study of SOP. As illustrations, we adapt to SEP, VEP and VVI enhanced notions of relative Pareto efficient solutions introduced in set optimization by Bao and Mordukhovich [25] and derive from known or new optimality conditions for various efficient solutions of SOP similar results for solutions of SEP, VEP and VVI.

In this paper, we also introduce the concept of quasi weakly efficient solutions for the above problems and divide all efficient solutions under consideration into the *Pareto-type* group containing Pareto efficient, primary relative efficient, intrinsic relative efficient, quasi relative efficient solutions and the *weak Pareto-type* group containing quasi weakly efficient, weakly efficient, strongly efficient, positive properly efficient, Henig global properly efficient, Henig properly efficient, super efficient and Benson properly efficient solutions. The main difference between efficient solutions of these two groups is that necessary conditions for weak Pareto-type solutions become sufficient under additional convexity assumptions while necessary conditions for Pareto-type efficient solutions we are dealing with here do not even in the finite-dimensional setting. We would also like to mention that all efficient solutions in the weak Pareto-type group, except quasi weakly efficient solutions, can be characterized in many cases by disjointness of a set and an open convex cone or by disjointness of a set and convex open cones from some parametrized family; therefore, we can consider them as weakly efficient solutions w.r.t. appropriately chosen convex cones with nonempty interior (hence the terminology "weak Pareto-type"). Our approach to these efficient solutions is a specialization of the unified approach proposed in the author's recent work [26] in which various kinds of efficient solutions have been characterized by disjointness of a set and an open non-necessarily convex cone.

We will use necessary conditions for Pareto-type efficient solutions and weakly efficient solutions of SOP in the Asplund space setting established with the help of advanced tools of variational analysis by Bao and Mordukhovich in [25], necessary and sufficient conditions for weakly efficient solutions and necessary conditions for strongly efficient solutions and properly efficient solutions of a general constrained set-valued optimization problem (for short, GSOP) in the Banach space setting established with the use of scalarization techniques by El Abdouni and Thibault [27] and by the author in [26]. We will prove some new necessary conditions for quasi weakly efficient solutions, super efficient solutions of SOP and sufficient conditions for quasi weakly efficient solutions, strongly efficient solutions, properly efficient solutions of SOP. The optimality conditions formulated here are expressed in terms of the Ioffe approximate coderivative and normal cone in the Banach space setting and in terms of the Mordukhovich coderivative and normal cone in the Asplund space setting.

The paper is organized as follows. In Section 2, we recall concepts of normal cone and coderivative, the sequential normal compactness of a set, the pseudo-Lipschitzity and the partial sequential normal compactness of a set-valued map. Section 3 is devoted to concepts of efficient points for sets. In Section 4, we recall some known optimality conditions for efficient solutions of SOP and establish some new ones. Section 5 is devoted to the problems SEP, VEP and VVI. Recalling some known concepts of efficient solutions of these problems and introducing some new ones, we establish an equivalence between efficient solutions of SOP and SEP and derive optimality conditions for efficient solutions of SEP, VEP and VVI from the results formulated for efficient solutions of SOP in Section 4.

2. Some tools from variational analysis

For the convenience of the reader we repeat the relevant material from [28–40] without proofs, thus making our exposition self-contained.

Throughout the paper, unless otherwise specified, X and Y are Banach spaces with their duals X^* and Y^*, respectively. We use the same notation $\|.\|$ for the norms in all these spaces. The closed unit ball and the open unit ball in any space, say X, are denoted by $\mathbb{B}_X$ and $\mathring{\mathbb{B}}_X$; we omit the subscript X when no confusion occurs. For a nonempty set $A \subset X$, $\operatorname{int} A$ and $\operatorname{cl} A$ stand for the interior and closure of A and $\operatorname{cone} A := \{ta : t \in \mathbb{R}_+, a \in A\}$, where $\mathbb{R}_+ = [0, \infty[$. Further, $d(x; A)$ is the distance from x to A. Recall that a Banach space is *Asplund* if each of its separable subspace has a separable dual. This class of spaces has been comprehensively investigated in geometric theory of Banach spaces and has been largely employed in variational analysis; see, e.g. [37,38]. Examples of Asplund spaces are the Banach spaces $\mathbb{R}^n$, $L^p_{[0,1]}$ and l^p $(1 < p < \infty)$.

Assume that $g : X \mapsto \mathbb{R} \cup \{\infty\}$ is a function and $F : X \mapsto 2^Y$ is a set-valued map (for the sake of convenience we assume that $F(x)$ is nonempty for all $x \in X$). The domain, epigraph of g and the graph of F are the sets $\operatorname{dom} g = \{x \in X \mid g \text{ is finite at } x\}$, $\operatorname{epi} g = \{(x, t) \in X \times \mathbb{R} \mid g(x) \le t\}$ and $\operatorname{gr} F = \{(x, y) \in X \times Y \mid y \in F(x)\}$, respectively.

Let us begin with recalling the construction for defining normal cones and coderivatives. Recall that given a locally Lipschitz function g and $x \in \operatorname{dom} g$, the *Ioffe approximate subdifferential* of g at x [29–31] is the set

$$\partial_A g(x) = \bigcap_{L \in \mathcal{F}} \limsup_{(\varepsilon, y) \to (0^+, x)} \partial^-_\varepsilon g_{y+L}(y),$$

where $\mathcal{F}$ is the collection of all finite-dimensional subspaces of X, $g_{y+L}(u) = g(u)$ if $u \in y + L$ and $g_{y+L}(u) = +\infty$ otherwise, for $\varepsilon \ge 0$

$$\partial^-_\varepsilon g_{y+L}(y) = \{x^* \in X^* \mid x^*(v) \le \varepsilon\|v\| + \liminf_{t \to 0^+} t^{-1}[g_{y+L}(y + tv) - g_{y+L}(y)],\ \forall v \in X\}.$$