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Optimal control of multivalued quasi variational inequalities

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ABSTRACT

Optimal control of various variational problems has been an area of active research. On the other hand, in recent years many important models in mechanics and economics have been formulated as multi-valued quasi variational inequalities. The primary objective of this work is to study optimal control of the general nonlinear problems of this type. Under suitable conditions, we ensure the existence of an optimal control for a quasi variational inequality with multivalued pseudo-monotone maps. Convergence behavior of the control is studied when the data for the state quasi variational inequality is contaminated by some noise. Some possible applications are discussed.

1. Introduction

Optimal control of partial differential equations (PDEs) and variational inequalities is an expanding and vibrant branch of applied mathematics that has found numerous applications. Although the theory and computational techniques for optimal control for PDEs and variational inequalities have been studied for quite some time now, it seems that there are still many unanswered questions and many interesting ideas are still in the making. Nonetheless, in recent years many important applied models have motivated the study of optimal control for more general variational problems.

In this work, we study the optimal control of quasi variational inequalities with multivalued maps. A quasi variational inequality is an important generalization of variational inequalities which was introduced by Bensoussan and Lions [1] in connection with impulse control. Since then the theory of quasi variational inequalities has emerged as one of the most promising branches of pure, applied, and industrial mathematics. This theory gives us a powerful mathematical apparatus for studying a wide range of problems arising in diverse fields such as structural mechanics, elasticity, economics, optimization, financial mathematics, and others (see [2–8]). However, despite the fact that there are many important models leading to quasi variational inequalities, the corresponding optimal control problem is largely unexplored. This is partly due to the fact that the theory for maps more general than monotone maps, and fast and reliable computing techniques for quasi variational inequalities are still being developed (see [9,10]).

Nonetheless, it was Dietrich [11] who took the first step in the study of optimal control of quasi variational inequalities. By assuming that the underlying constraint set has a special structure, he derived necessary optimality conditions for the control problem. He introduced a smooth gap function (see also [9]) to convert the quasi variational inequality into an

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optimization problem, and by means of a penalty method, derived necessary optimality conditions. In another important contribution [12], the authors gave an existence result for a control problem for a quasi variational inequality with single-valued monotone maps.

Most of the related work on optimal control has been done in the context of variational inequalities. Mignot [13], in one of the earlier works on optimal control of variational inequalities, showed that the control-to-solution map is, in general, non-smooth. He presented a novel approach to obtain optimality conditions by introducing the conical derivatives. Barbu, in his seminal works, tackled the optimal problem by using a penalization technique and obtained new generalized optimality conditions. An excellent reference to some of Barbu's works and other earlier developments is the monograph [14]. Shi [15] employed the penalty method and the celebrated Ekeland's variational principle to give optimality conditions for a non-convex control problem in which the system is governed by a general variational inequality with a strongly monotone operator. Ndoutoume [16] presented a general approach to obtain optimality conditions by using epi-convergence of second-order differential quotients, and as an application, gave first-order necessary optimality conditions for a non-smooth control problem governed by variational inequalities. Optimal control for variational inequalities with single-valued pseudo-monotone maps is studied by Patrone [17]. One of the most successful and numerically efficient techniques to obtain the optimality system for the optimal control problem for variational inequalities is the so called primal–dual active set method proposed by Ito and Kunisch [18]. The authors used the augmented Lagrangian method to treat the infinite-dimensional optimization problem and showed that the conditions of convergence of the method are fulfilled in interesting applications and gave promising numerical results. Some of these ideas have been strengthened in an important contribution by Hintermüller [19], who used the notion of complementarity functions to penalize the lower-level problem and to obtain an optimality system. For details and related studies, the interested reader is referred to the excellent monographs by Ito and Kunisch [20] and by Gunzburger [21].

In this work, we study the optimal control of quasi variational inequalities with multivalued monotone and pseudo-monotone maps. The motivation to investigate optimal control problem for multi-valued quasi variational inequalities stems from an interesting recent paper by Kano et al. [22]. In this work, by means of several useful applications, the authors showed the necessity to explore multivalued quasi variational inequalities. Besides the existence of an optimal control for quasi variational inequalities with pseudo-monotone maps, we also investigate the behavior of the control when the data for the underlying quasi variational inequality is contaminated by some noise.

The organization of this paper is as follows. In Section 2 we describe the control problem that is the subject of this paper. In Section 3 we collect some of the basic notions and recall some existence results for quasi variational inequalities. Section 4 gives existence results for an optimal control problem for a multivalued quasi variational inequality. Convergence analysis for the optimal control problem when the underlying quasi variational inequality is subjected to some perturbation is given in Section 5. In Section 6 we give applications of our results to implicit obstacle problems, hemi quasi variational inequalities, and an Elasto-plastic torsion model. Some concluding remarks are given in Section 7.

2. Problem formulation

Let $\mathscr{X}$ be a real reflexive Banach space and let $\mathscr{X}^*$ be its topological dual. We specify the duality pairing between $\mathscr{X}$ and $\mathscr{X}^*$ by $\langle \cdot, \cdot \rangle$, whereas $\|\cdot\|_{\mathscr{X}}$ stands for the norm in $\mathscr{X}$ as well as the associated norm in $\mathscr{X}^*$. Let $\mathscr{C}$ be a nonempty, closed, and convex subset of $\mathscr{X}$, and let $\mathscr{K} : \mathscr{C} \rightrightarrows \mathscr{C}$ be a multivalued map such that for every $v \in \mathscr{C}$, the set $\mathscr{K}(v)$ is a nonempty, closed, and convex subset of $\mathscr{C}$. Let $\mathscr{F} : \mathscr{X} \rightrightarrows \mathscr{X}^*$ be a given multivalued map, and let $f \in \mathscr{X}^*$ be arbitrary. The domain and the graph of $\mathscr{F}$ are given by

$$\mathscr{D}(\mathscr{F}) := \{x \in \mathscr{X} : \mathscr{F}(x) \neq \emptyset\}$$
$$\mathscr{G}(\mathscr{F}) := \{(x, w) : x \in \mathscr{D}(\mathscr{F}),\ w \in \mathscr{F}(x)\},$$

respectively. We specify the strong convergence by $\to$ and the weak convergence by $\rightharpoonup$.

Let us introduce the following multivalued quasi variational inequality: find $x \in \mathscr{K}(x)$ such that for some $w \in \mathscr{F}(x)$, we have

$$\langle w - f, z - x\rangle \geq 0, \quad \text{for every } z \in \mathscr{K}(x). \tag{1}$$

In the following, the notation $\mathscr{S}(\mathscr{F}, \mathscr{K}, f)$ will represent the set of all solutions of (1).

The above quasi variational inequality includes many important problems of interest as particular cases. For instance, if the map $\mathscr{F}$ is single-valued, then (1) recovers the following quasi variational inequality: find $x \in \mathscr{K}(x)$ such that

$$\langle \mathscr{F}(x) - f, z - x\rangle \geq 0, \quad \text{for every } z \in \mathscr{K}(x).$$

The above problem was introduced by Bensoussan and Lions [1] in connection with an impulse control problem, and a general treatment was given by Mosco [23], among others. If additionally $\mathscr{K}(x) = \mathscr{C}$ for every $x \in \mathscr{C}$, then (1) recovers the following celebrated variational inequality: find $x \in \mathscr{C}$ such that

$$\langle \mathscr{F}(x) - f, z - x\rangle \geq 0, \quad \text{for every } z \in \mathscr{C}.$$