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On coderivatives and Lipschitzian properties of the dual pair in optimization

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ABSTRACT

In this paper, we apply the concept of coderivative and other tools from the generalized differentiation theory for set-valued mappings to study the stability of the feasible sets of both the primal and the dual problem in infinite-dimensional linear optimization with infinitely many explicit constraints and an additional conic constraint. After providing some specific duality results for our dual pair, we study the Lipschitz-like property of both mappings and also give bounds for the associated Lipschitz moduli. The situation for the dual shows much more involved than the case of the primal problem.

1. Introduction

This paper deals with the following linear optimization problem

$$P: \quad \begin{array}{ll} \text{Sup} & \langle \overline{c}^*, x\rangle \\ \text{s.t.} & \langle a_t^*, x\rangle \le \overline{b}_t, \quad t\in T, \\ & x \in Q, \end{array} \tag{1}$$

where T is an arbitrary index set, possibly infinite, Q is a convex cone in a real Banach space X, $\overline{c}^*$ and a_t^*, $t \in T$, belong to the topological dual of X, denoted by X^*, and $\overline{b}_t$, $t \in T$, are real numbers. P is an infinite-dimensional optimization problem with possibly infinitely many linear inequality constraints (depending on the cardinality of T).

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Problems of this type have relevant applications in science and technology. A number of them are reported in [1,2], where the reader can find comprehensive overviews of *infinite-dimensional* and *semi-infinite optimization*, respectively. See also [3], which is confined to the so-called *continuous problem* (when the index set T is a compact Hausdorff space and the functions $t \mapsto a_t^*$ and $t \mapsto \overline{b}_t$ are continuous).

We assume that Q is *closed* and that the set $\{a_t^*,\ t \in T\} \subset X^*$ is fixed, arbitrary, and *bounded* for the dual norm in X^* defined by

$$\|x^*\| := \sup\left\{\langle x^*, x\rangle : \|x\| \leq 1\right\}.$$

(If no confusion arises, we use the same notation $\|\cdot\|$ for the given norm in X and the corresponding dual norm in X^*.)

As a consequence of the boundedness assumption and the generalized Cauchy–Schwarz inequality, we have that, for every $x \in X$,

$$\langle a_{(.)}^*, x\rangle \in \ell_\infty(T),$$

where $\ell_\infty(T)$ is the real Banach space of all bounded functions on T with the supremum norm

$$p \in \ell_\infty(T) \to \|p\|_\infty := \sup_{t\in T} |p_t|.$$

The subscript ∞ in the norm symbol will be omitted if no confusion arises. When the index set T is compact and the functions $a_{(.)}^*$ are continuous on T, we may substitute $\ell_\infty(T)$ by the space $\mathcal{C}(T)$ of continuous functions over a compact set.

By means of the linear mapping $A : X \to \ell_\infty(T)$ defined as $Ax := \langle a_{(.)}^*, x\rangle$, the problem P can be reformulated as

$$P: \quad \begin{array}{ll} \text{Sup} & \langle \overline{c}^*, x\rangle \\ \text{s.t.} & \begin{array}{l} Ax \leq \overline{b}, \\ x \in Q. \end{array} \end{array} \tag{2}$$

Here $\overline{b} = \left(\overline{b}_t\right)_{t\in T}$. Thanks to the boundedness of $\{a_t^*,\ t \in T\}$, the linear operator A is bounded, and so continuous, as

$$\|A\| = \sup_{\|x\|\leq 1} \|Ax\| = \sup_{\|x\|\leq 1} \sup_{t\in T} |\langle a_t^*, x\rangle| \leq \sup_{\|x\|\leq 1} \sup_{t\in T} \left\|a_t^*\right\| \ \|x\| = \sup_{t\in T} \left\|a_t^*\right\|.$$

If X is reflexive, associated with each $t \in T$, there exists some $x_t \in X$ such that $\|x_t\| = 1$ and satisfying $\langle a_t^*, x_t\rangle = \left\|a_t^*\right\|$; this fact leads to $\|A\| = \sup_{t\in T} \left\|a_t^*\right\|$.

The problem P is called *primal* as it has an associated *dual* problem D defined as follows:

$$D: \quad \begin{array}{ll} \text{Inf} & \langle \mu, \overline{b}\rangle \\ \text{s.t.} & \begin{array}{l} A^*\mu \in \overline{c}^* - Q^\circ, \\ \mu \geq 0, \end{array} \end{array}$$

where $\mu \in \ell_\infty(T)^*$, $A^* : \ell_\infty(T)^* \to X^*$ is the adjoint operator of A, i.e.

$$\left\langle A^*\mu, x\right\rangle = \langle \mu, Ax\rangle, \quad \text{for every } \mu \in \ell_\infty(T)^* \text{ and every } x \in X,$$

and Q° is the dual cone of Q

$$Q^\circ := \{q^* \in X^* : \langle q^*, q\rangle \leq 0 \text{ for all } q \in Q\}.$$

This dual problem falls in the duality model introduced by Kretschmer in [4] and it is developed here at an intermediate level of generality between the approaches in [5,6]. Anderson and Nash have given a detailed account of this theory in [1, Chapter 3]. In fact, our pair of dual problems P and D are particular instances of problems IP and IP^* in [1, pp. 38 and 39], respectively. Here, A is a continuous linear mapping between X and $\ell_\infty(T)$ with respect to the norm topologies, but Proposition 5 in [1, p. 37] applies to guarantee that our dual pair falls in the model studied in the book [1, Section 3.3]. Actually, the theory in [1, Section 3.3] is built on a *reflexive* context (dual pairs of vector spaces), but the reflexivity is required only to guarantee that the dual of the dual problem IP^*, i.e. IP^{**}, is identical to IP. Therefore, the reflexivity assumption has no influence in the arguments used in the proofs when this second dual IP^{**} is not involved.

The dual objects we study in the paper are the associated *feasible sets*

$$F_P := \left\{x \in X : Ax \leq \overline{b} \text{ and } x \in Q\right\},$$

and

$$F_D := \left\{\mu \in \ell_\infty(T)^* : A^*\mu \in \overline{c}^* - Q^\circ \text{ and } \mu \geq 0\right\},$$

the *optimal values*

$$v_P := \sup_{x\in F_P} \left\langle \overline{c}^*, x\right\rangle \quad \text{and} \quad v_D := \inf_{\mu\in F_D} \left\langle \mu, \overline{b}\right\rangle,$$