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Complete characterizations of local weak sharp minima with applications to semi-infinite optimization and complementarity

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ABSTRACT

In this paper, we identify a favorable class of nonsmooth functions for which local weak sharp minima can be completely characterized in terms of normal cones and subdifferentials, or tangent cones and subderivatives, or their mixture in finite-dimensional spaces. The results obtained not only extend previous ones in the literature, but also allow us to provide new types of criteria for local weak sharpness. Applications of the developed theory are given to semi-infinite programming and to a new class of semi-infinite complementarity problems.

1. Introduction

This paper is mainly devoted to the study and complete characterizations of local weak sharp minima and their applications to problems of semi-infinite optimization and semi-infinite complementarity in finite-dimensional spaces.

Given an extended-real-valued function $f:\mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R}\cup\{\infty\}$ and a point $x \in \mathbb{R}^n$ with $f(x) < \infty$, recall that x is *local weak sharp minimum* of f if there exist positive scalars η and δ such that

$$\eta\,\mathrm{dist}\big(z, L_f(x)\big) \le f(z) - f(x) \quad \text{for all } z \in B(x,\delta), \tag{1.1}$$

where $B(x,\delta)$ is the closed ball with center x and radius $\delta > 0$, where

$$L_f(x) := \big\{z \in \mathbb{R}^n \,|\, f(z) = f(x)\big\},$$

is the level set of f at x, and where $\mathrm{dist}(x, A)$ is the distance function from x to a given set $A \subset \mathbb{R}^n$ defined by

$$\mathrm{dist}(x,A) := \inf_{y\in A} \|x-y\|$$

with $\|\cdot\|$ standing for the standard Euclidean norm. Definition (1.1) clearly implies that x is a local minimum of f.

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The notion of weak sharp minima was introduced by Ferris in [1] as a generalization of sharp minima due to Polyak [2] to include the possibility of non-unique solutions. During the last two decades the study of weak sharp minima has drawn much attention motivated by its importance in the treatment of sensitivity analysis (see, e.g., [3,4]) and of convergence analysis for a wide range of optimization algorithms; we refer the reader to [5–11] and the bibliographies therein. Roughly speaking, efficient conditions for weak sharp minima obtained in these papers via generalized differentiation can be classified into two types: *primal* conditions and *dual* conditions. The former involve tangent cones and directional derivatives, while the latter employ normal cones and subdifferentials.

Observe that necessary and sufficient conditions for local weak sharp minima were established in two special cases. The first case concerns the situation when x is a strict local minimum. Then definition (1.1) reduces, by shrinking δ if necessary, to

$$\eta\|z-x\| \le f(z)-f(x) \quad \text{for all } z \in B(x,\delta), \tag{1.2}$$

which is often referred to as local sharp minimum and is also called strongly unique local minimum; cf. [12,2]. In this case it is not difficult to verify (see, e.g., [3, Chapter 3]) that (1.2) holds if and only if $\mathrm{d}f(x)(w) > 0$ for all nonzero $w \in \mathbb{R}^n$ via the subderivative of f defined in Section 2. Second, when the problem data are convex Burke and Ferris [5] provided several primal and dual characterizations of weak sharp minima and studied its impact to convex programming and convergence analysis in finite-dimensional setting; this was further extended by Burke and Deng [13] to infinite dimensions. Furthermore, close relationships between weak sharp minima, linear regularity, metric regularity, and error bound were exploited in [14,15]. The recent paper [16] considers weak sharp minima for convex constrained optimization problems on Riemannian manifolds, containing also new characterizations for the case of conventional convex problems in finite-dimensional spaces.

In the general case, however, the nonconvexity of f and the non-uniqueness of solutions give rise to a lot of complications that invalidate classical techniques. To circumvent these difficulties, several approaches have been proposed. In particular, Wu and Ye [17] obtained dual sufficient conditions for global weak sharp minima in terms of an abstract subdifferential, a fairly general concept unifying most of specific subdifferentials useful in variational analysis. In [18], Ng and Zheng presented primal sufficient conditions for a proper lower semicontinuous function on a Banach space to have global weak sharp minima by using various kinds of lower generalized derivatives.

It is worth noting that the notion of weak sharp minima defined in (1.1) underlines a first-order growth of the objective function away from the level set $L_f(x)$. Meanwhile, weak sharp minima of higher order growth are also of interest in parametric optimization, because it can be used to establish Hölder continuity properties of solution mappings. In particular, weak sharp minima of order two was studied by Bonnans and Ioffe [19] in the case when f is a pointwise maximum of twice continuously differentiable convex functions. Sufficient conditions for weak sharp minima of order $m \ge 1$ for nonconvex functions in finite dimensions were obtained by Studniarski and Ward [20] via the limiting normal cone by Mordukhovich and a certain extension of the regular tangent cone by Clarke.

Observe that, except for the two cases mentioned above and some particular situations, most of the conditions obtained for local weak sharp minima are either necessary or sufficient but not both. A natural and important question arises about the possibility to establish necessary and sufficient conditions for local weak sharp minima when f is not necessarily convex and x is not restricted to be a strict local minimum. An significant step in this direction was made by Zheng and Yang [21] who derived characterizations of local weak sharp minima for semi-infinite programming by exploiting the special structure of functions involved therein. Newer dual results in this direction have been recently obtained by Zheng and Ng [22,23]; see more discussions in Section 4.

The main purpose of this paper is to obtain efficient characterizations of local weak sharp minima in the general nonconvex framework of nonsmooth functions f in (1.1) and then to apply them to important classes of optimization-related problems. Our necessary and sufficient conditions are not only extend the aforementioned ones to broader classes of problems but also offer verifiable criteria of new types to characterize local weak sharp minima in both convex and nonconvex settings.

To achieve our goals, we introduce a new class of nonsmooth functions, called *inf-differentiable* functions, which are certainly of their independent interest. It is shown below that this class is sufficiently broad to cover a number of special classes of functions overwhelmingly encountered in variational analysis and optimization. Among those, besides the classical classes of smooth and convex functions, we particularly mention semidifferentiable functions, lower-C^1 functions, and functions given by parametric integrals with respect to finite measures over compact sets. The main results of this paper provide primal, dual, and mixed characterizations of local weak sharp minima for inf-differentiable functions in finite-dimensional spaces. These results enable us to fully characterize weak local sharp minima of semi-infinite programs entirely in terms of their initial data and to derive necessary and sufficient conditions for local error bounds of residuals for a new class of semi-infinite complementarity problems.

The rest of the paper is organized as follows. Section 2 collects some preliminaries from generalized differentiation widely used in the sequel. In Section 3 we introduce inf-differentiable functions and establish their relationships with other favorable classes of functions in variational analysis and optimization. Section 4 is devoted to characterizing local weak sharp minima for inf-differentiable functions. In Sections 5 and 6 we illustrate applications of the developed theory to important classes of problems in semi-infinite programming and semi-infinite complementarity, respectively. The final Section 7 presents concluding remarks and discussions on further research.