Nonlinear Analysis 75 (2012) 1719–1736

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Convex analysis in financial mathematics

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ARTICLE INFO

Communicated by Ravi Agarwal

Keywords:
Convex analysis
Duality
Financial mathematics

ABSTRACT

Using the language of convex analysis, we describe key results in several important areas of finance: portfolio theory, financial derivative trading and pricing and consumption based asset pricing theory. We hope to emphasize the importance of convex analysis in financial mathematics and also draw the attention of researchers in convex analysis to interesting issues in financial applications.

1. Introduction

Concave utility functions and convex risk measures play crucial roles in economic and financial problems. The use of concave utility functions can be traced back to Bernoulli, who posed and solved the St. Petersburg wager problem. Since then, this method has been used for characterizing rational market participants. Markowitz used variation as a risk measure in his pioneering work on portfolio theory [1]. This is a quadratic risk measure that has played a prominent role in subsequent related work such as the capital market asset pricing model and the Sharpe ratio for evaluating investment performance. Artzner et al. [2] proposed the concept of coherent measure based on the practices of risk control for large clearing houses. This was later generalized to a convex risk measure in [3–6]. Moreover, in the general equilibrium theory of economics, convex sets also play key roles in describing the production, consumption and their exchange. The essential roles of these convex objects made convex analysis an indispensable tool in dealing with problems in finance. The purpose of this paper is to highlight the crucial role of convex analysis in financial research by using the convex analysis language to describe key results in several important areas of finance: portfolio theory, financial derivative trading and pricing and consumption based asset pricing theory.

To set the stage, we first lay out a discrete model for the financial market in Section 2. This largely follows the notation in [7]. The discrete model avoids much technical difficulty associated with the continuous model of financial markets and allows us to concentrate more on the principles. After describing the model, we explain the concept of arbitrage and the no arbitrage principle. This is followed by the important fundamental theorem of asset pricing in which the no arbitrage condition is characterized by the existence of martingale measures (also known as risk-neutral measures). The proof of this theorem relies on the convex separation theorem, which gives us a first taste of the importance of convex analysis tools. Next, we discuss how to use utility functions and risk measures to characterize the preference of market agents in Section 3. We lay out assumptions that are commonly imposed on utility functions and risk measures and give their financial explanation. It is interesting to see that the no arbitrage principle can also be characterized with a class of utility functions in terms of the utility being finite. Once the preliminary material is in place, we turn to discuss a number of key results in several important areas of finance systematically using convex analysis.

Section 4 is about portfolio theory. We discuss the Markowitz portfolio theory first. The related capital asset pricing model follows. We show that mathematically both of them are related to quadratic optimization with linear constraints, the simplest form of convex programming, and enjoy explicit solutions. The Sharpe ratio, an important measure for performances of different investment methods, is then derived as a consequence. Finally, we touch upon the capital growth

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0362-546X/$ – see front matter
doi:10.1016/j.na.2011.05.052

model in which maximum capital growth is the goal. We illustrate the general pattern and emphasize that the optimal solution to the capital growth model is not stable.

Section 5 deals with the issue of pricing financial derivatives. We use simple models to illustrate the idea of the prevailing Black–Scholes replicating portfolio pricing method [8,9] and related Cox–Ross [10,11] risk-neutral pricing method for financial derivatives. A widely held belief about these methods is that they are independent of the individual market player's preference, thereby providing a uniformly applicable pricing mechanism for financial derivatives. However, we show that the replicating portfolio pricing method is a special case of portfolio optimization by maximizing a particular kind of concave utility functions and the risk-neutral measure is a natural by-product of solving the dual problem. Thus, using the Black–Scholes option pricing mechanism and the related risk-neutral measure pricing method is, in fact, implicitly accepting a utility function along with other assumptions associated with these methods such as infinite leverage and one can use high frequency trading to maintain the replicating portfolio without paying much transaction cost etc. These observations point to necessary cautions when using the Black–Scholes and Cox–Ross pricing methods. More importantly, understanding these methods are, in fact, related to utility optimization naturally leads to the consideration of their sensitivity. It turns out these pricing methods are rather sensitive to model perturbations: a small deviation from the perceive the market model may well leads to the perceived arbitrage position constructed according to the theoretical market model to become pure losing positions. These theoretical flaw were also reflected in the real markets through the financial crises caused by the collapse of Long Term Capital Management in 1998 (see [12]) and the recent financial crisis of 2008. The unsatisfactory effect of the prevailing pricing and trading mechanism for financial derivatives calls for alternative ways of pricing and trading financial derivatives. It seems that the time tested utility optimization method is still highly relevant and a financial derivative market in which different players using different approaches is reasonable. One of such method emphasizing the robustness of the pricing and trading is discussed with tests conducted using real historical market data. Convex analysis plays a crucial role in this robust pricing and trading method.

In Section 6 we discuss a consumption based pricing model in which the pricing of financial assets are directly determined through the interaction of production, consumption and saving in a competitive market. While the idea of competitive market determines the price can be traced back to Adam Smith's invisible hand, it was L. Walras who first attempted a mathematical model for a general equilibrium in his 1877 treaties "Elements of Pure Economics". Rigorous formulation of the model and the proof of the existence of an equilibrium pricing was achieved by Arrow and Debreu in the 1950's [13]. We choose to present the influential Lucas' model [14]. In analyzing this model, convex analysis is combined with dynamical programming. We also briefly discuss the recent developments on extending the Lucas model to model term structure of interest rates.

This is an attempt to illustrate the importance of convex analysis in financial problems. We selected examples using one period, multi-period and dynamic programming models to emphasize that tools in convex analysis, in particular, convex duality is indispensable in dealing with financial problems with different degrees of sophistication. We also hope this will draw attentions of researchers in convex analysis to the many challenges arise in financial applications.

2. A discrete model for the financial markets

We will use a finite set Ω to represent all possible economic states and assume that the natural probability of each state is described by a probability measure P on the power set of Ω. We assume that $P(\omega) > 0$ for all $\omega \in \Omega$. Let $RV(\Omega)$ be the finite dimensional Hilbert Space of all random variables defined on Ω, with inner product

$$\langle \xi, \eta \rangle = \mathbf{E}^P(\xi\eta) = \int_\Omega \xi\eta \mathrm{d}P = \sum_{\omega\in\Omega} \xi(\omega)\eta(\omega)P(\omega).$$

When there is no risk of ambiguity we will omit the superscript P and write $\mathbf{E}(\xi\eta) = \mathbf{E}^P(\xi\eta)$. For $\xi \in RV(\Omega)$ we use $\xi > 0$ to signal $\xi(\omega) \geq 0$ for all $\omega \in \Omega$ and at least one of the inequality is strict. We consider a discrete model in which trading action can only take place at $t = 0, 1, 2, \ldots$.

Let $\mathcal{F} = \{\mathcal{F}_t \mid t = 0, 1, \ldots\}$ be an *information system* of σ-algebras of subsets of Ω, that is,

$$\sigma(\{\Omega\}) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_1 \subset \cdots \quad \text{and} \quad \cup_{t=0}^{\infty} \mathcal{F}_t = \sigma(\Omega).$$

Here, for each $t = 0, 1, \ldots, \mathcal{F}_t$, represents available information at time t. Thus, an information system represents a framework in which we never loss any information. Moreover, to begin with at $t = 0$ we know nothing and our knowledge increase with time t. We will often consider finite period economy where $t = 0, 1, \ldots, T$. In this case $\mathcal{F}_T = \sigma(\Omega)$. The triple $(\Omega, \mathcal{F}, P)$ is a way to model the gradually available information about the economy.

Proposition 2.1. *In a finite period economy. For each $t = 0, 1, \ldots, T$, let P_t be the set of atoms of the σ-algebra $\mathcal{F}_t$. Then*

(a) *for each t, P_t is a partition of Ω, and*
(b) *$\mathcal{F}$ is an information system if and only if*

$$\{\Omega\} = P_0 \prec P_1 \prec \cdots \prec P_T = \{\{\omega\} : \omega \in \Omega\}.$$

Here $P \prec Q$ signifies that Q is a refinement of P.