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Global existence and blow-up for the generalized sixth-order Boussinesq equation

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ABSTRACT

In this paper we prove local well-posedness in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ for the generalized sixth-order Boussinesq equation $u_{tt} = u_{xx} + \beta u_{xxxx} + u_{xxxxxx} + (|u|^\alpha u)_{xx}$. Our proof relies in the oscillatory integrals estimates introduced by Kenig et al. (1991) [14]. We also show that, under suitable conditions, a global solution for the initial value problem exists. In addition, we derive the sufficient conditions for the blow-up of the solution to the problem.

1. Introduction

We study the initial value problem (IVP) for the generalized sixth-order Boussinesq equation

$$\begin{cases} u_{tt} = u_{xx} + \beta u_{xxxx} + u_{xxxxxx} + (|u|^\alpha u)_{xx}, & x \in \mathbb{R},\ t \geq 0, \\ u(0,x) = \varphi(x);\ u_t(0,x) = \psi_{xx}(x), \end{cases} \tag{1.1}$$

where $\beta = \pm 1$ and $\alpha > 0$.

Eq. (1.1) was derived in the shallow fluid layers and nonlinear atomic chains [1,2]. Maugin in [3] also proposed (1.1) in modeling the nonlinear lattice dynamics in elastic crystals. Feng et al. [4] studied the solitary waves and their interactions for the above Eq. (1.1).

In [5], Esfahani and Farah studied the local well-posedness for the initial value problem associated to (1.1) with quadratic nonlinearity. In this particular case, they proved local well-posedness for (1.1) in the classical Sobolev space $H^s(\mathbb{R})$ with $s > -1/2$. In the present paper our main aim is to study this issue for (1.1) with power-law-type nonlinearity $|u|^\alpha u$.

One can observe that by ignoring the sixth-order term in (1.1), one obtains the classical generalized Boussinesq equation

$$u_{tt} = u_{xx} + \beta u_{xxxx} + (|u|^\alpha u)_{xx}, \tag{1.2}$$

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which arises in the modeling of nonlinear strings. Depending on the sign of β the Eq. (1.2) represents two distinct problems. Indeed, the case $\beta = 1$ is called a "bad" Boussinesq equation and only soliton type solutions are known. Concerning the case $\beta = -1$, several local well-posedness results have been obtained in the last years. Using Kato's abstract theory for the quasilinear evolution equation, Bona and Sachs [6] showed local well-posedness for a smooth nonlinearity f and initial data $u(x, 0) = \phi(x) \in H^{s+2}(\mathbb{R})$ and $u_t(x, 0) = \psi_x(x) \in H^{s+1}(\mathbb{R})$ with $s > 1/2$. Tsutsumi and Matahashi [7] established a similar result for the initial data $\phi \in H^1(\mathbb{R})$ and $\psi_x = \chi_{xx}$ with $\chi \in H^1(\mathbb{R})$. These results were significantly improved by Linares [8] who proved that (1.2) is locally well-posed in the case $0 < \alpha < 4$ and $\phi \in L^2(\mathbb{R})$, $\psi \in H^{-1}(\mathbb{R})$. The main tool used in his argument was the Strichartz estimates satisfied by solutions of the linear problem. Finally, using the Fourier restriction norm approach, Farah [9] proved local well-posedness for quadratic nonlinearity (so-called "good" Boussinesq equation), $\phi \in H^s(\mathbb{R})$, $\psi \in H^{s-1}(\mathbb{R})$ and $s > -1/4$.

In our case, to show the local existence of a solution for (1.1), the idea of our proof is to exploit the dispersive character of Eq. (1.1) introduced by higher order derivatives. In other words, we shall use the so-called L^p–L^q estimates of Strichartz type that arise in the study of Schrödinger equation [10]. Strichartz's result has been generalized and its proof simplified in the works of Marshall [11], Pecher [12], and Ginibre and Velo [13], who proved that the solution of the one-dimensional Schrödinger equation has the property

$$\left(\int_{\mathbb{R}} \|e^{it\Delta}u_0\|_{L^\infty}^4 \, dt\right)^{1/4} \leq C\|u_0\|_{L^2},$$

where $e^{it\Delta}u_0$ denotes the solution of the linear Schrödinger equation with initial data u_0.

In [14], Kenig, Ponce and Vega extended this result to a general class of high order dispersive equations

$$\begin{cases} u_t + iP(D)u = 0, & x, t \in \mathbb{R}, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.3}$$

where $D = (1/i)(\partial/\partial x)$ and $P(D)$ is defined via $(P(D)f)^\wedge(\xi) = P(\xi)\widehat{f}(\xi)$ and $\wedge$ is the Fourier transform. They showed that the solution $u(x, t)$ of problem (1.3) satisfies

$$\left(\int_{\mathbb{R}} \|D_x^{(\alpha-2)/4}u(\cdot, t)\|_{L^\infty}^4 dt\right)^{1/4} \leq C\|u_0\|_{L^2},$$

for some $\alpha > 2$ corresponding to the order the symbol $P(D)$.

Using the general setting introduced in [14] we obtain Strichartz type estimates satisfied by solutions of the linear problem associated to Eq. (1.1). Combining these inequalities and a contraction mapping argument we obtain the following $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ local well-posedness results.

Theorem 1.1. *If* $0 < \alpha < 4$ *then for all* $\phi \in L^2(\mathbb{R})$ *and* $\psi \in \dot{H}^{-1}(\mathbb{R})$ *there exist* $T > 0$ *and a unique solution* u *of* (1.1) *in* $[0, T]$ *with*

$$u \in C\left([0, T] : L^2(\mathbb{R})\right) \cap L^4\left([0, T] : L^\infty(\mathbb{R})\right)$$

and

$$D_x u \in L^\infty(\mathbb{R} : L^2([0, T])).$$

Theorem 1.2. *If* $0 < \alpha$ *then for all* $\phi \in H^1(\mathbb{R})$ *and* $\psi \in L^2(\mathbb{R})$ *there exist* $T > 0$ *and a unique solution* u *of* (1.1) *in* $[0, T]$ *with*

$$u \in C\left([0, T] : H^1(\mathbb{R})\right) \cap L^4\left([0, T] : L^\infty(\mathbb{R})\right)$$

and

$$D_x^2 u \in L^\infty(\mathbb{R} : L^2([0, T])).$$

We should mention that our local theory applies to both cases $\beta = \pm 1$ in Eq. (1.1). This situation also appears in the results of [5] and it is very different from the generalized Boussinesq equation (1.2), where the "good" and "bad" models are very distinct.

Our next aim is to give sufficient conditions for the existence of a global solution for problem (1.1). To do this, we employed methods introduced in [15] (see also [16]), by constructing suitable sets which are invariant under the flow generated by our Cauchy problem, and choosing an appropriate initial data. We also derive conditions for the blow-up of the solution to problem (1.1) by using a differential inequality for a functional of the solution to (1.1). For the non-positive energy, a blow-up result is proved by the potential well argument and the concavity method introduced by Payne and Sattinger [17] and Levine [18–20], see also [21,22].

The plan of this paper is the following. In Section 2, we prove some preliminary results. Linear estimates and local smoothing effects are proved in Section 3. The local existence theory is established in Section 4. Finally, the global well-posedness and blow-up results are treated in Section 5.