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Critical points of solutions to quasilinear elliptic problems

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ABSTRACT

We describe the set of critical points (points of vanishing gradient) associated to solutions of an important class of quasilinear elliptic problems with zero Dirichlet condition in planar domains. We show that the critical set is made up of finitely many isolated points and finitely many (regular) analytic Jordan curves. Further, we generalize the well-known result of Makar-Limanov, according to which the solution to the Poisson equation $\Delta u = 1$, with zero Dirichlet condition in a convex domain, has a unique critical point.

1. Introduction

Important mathematical models in engineering and physics often reduce to a boundary value problem of the type:

$$\begin{aligned} &\mathrm{L}[u] = f(u) && \text{in } \Omega, \\ &u = 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a planar domain, f is a real value function, and L is a quasilinear elliptic operator,

$$\mathrm{L}[u] \equiv a(\nabla u)\frac{\partial^2 u}{\partial x_1^2} + b(\nabla u)\frac{\partial^2 u}{\partial x_1 \partial x_2} + c(\nabla u)\frac{\partial^2 u}{\partial x_2^2}. \tag{2}$$

Several of the most studied second order PDEs fall into this category. Such is the case for example, of the inhomogeneous *A*-Laplacian equation (see [1])

$$\begin{aligned} &\operatorname{div}(A(|\nabla u|)\,\nabla u) = f(u) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

provided $A(s) + sA'(s) > 0$, $s > 0$. For instance, if $A(s) = |s|^{p-2}$ we obtain the inhomogeneous p-Laplacian equation which is elliptic whenever $p > 1$. Another interesting case is the well-known *prescribed mean curvature equation* which is obtained by setting $A(s) = \frac{1}{\sqrt{1+s^2}}$. On the other hand, if $A \equiv 1$ the A-Laplacian equation reduces to the well known semilinear case $\Delta u = f(u)$.

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In this paper we investigate the set of critical points (*critical set*) $\mathbb{K}$ of the solutions u to (1). Even in the case of the Poisson equation with constant inhomogeneous term, the set $\mathbb{K}$ has not been fully described, except for some few domains.

When Ω is convex and L is the linear elliptic operator $\mathrm{L}[u] = \Delta u$, several authors have proved that the critical set reduces to a single point, under Dirichlet conditions on the boundary, see for example [2–4], and the literature cited therein. Similar results have also been obtained for non-Dirichlet type conditions. Sakaguchi [5] for example, proved the existence of a unique critical point for Neumann and Robin problems. However, even for the semilinear case, few authors have considered nonconvex domains [6–8]. The goal of this paper is to extend some of the known results for semilinear operators in convex and nonconvex domains, to the quasilinear case.

Throughout this paper we shall assume that $\partial\Omega$ is smooth, the coefficients a, b and c are real analytic on $\mathbb{R}^2$, and the inhomogeneous term f is real analytic in $\mathbb{R}$. We also assume that f is increasing and that $f(0) > 0$. Furthermore L is uniformly elliptic, meaning the existence of a positive constant C such that for every $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$,

$$a(\xi)\,\eta_1^2 + b(\xi)\,\eta_1\,\eta_2 + c(\xi)\,\eta_2^2 > C\,|\eta|^2.$$

Within this framework existence of solutions cannot be taken for granted, but if a negative solution exists, it is unique and analytical (see for example [9]). The results of this investigation can be then summarized as follows:

1. If Ω is strictly convex then $\mathbb{K}$ is made up of exactly one point.
2. The set $\mathbb{K}$ is made up of finitely many isolated points and finitely many Jordan curves. Moreover, if there is any Jordan curve contained in the critical set, this curve must be analytic and the domain Ω cannot be simply connected.
3. If Ω is an annular domain with external boundary γ_E and internal boundary γ_I, both of positive curvature, then $\mathbb{K}$ is either a finite set of points, or is made up of exactly one Jordan curve with convex interior. Under additional symmetry assumptions we prove the nonexistence of Jordan critical curves on certain annular domains.

The first statement generalizes Makar-Limanov's result [10] (see also [2]), whereas the second and third statements generalize results obtained in our previous paper [7]. Several results in the same spirit than the ones provided in this paper have been previously established by different authors. For example Sakaguchi [11] proved uniqueness of the critical point for two different boundary value problems concerning the constant mean curvature equation on plane convex domains, while McCuan [12] surveys techniques and results on the quasiconcavity of solutions to more general quasilinear equations on convex domains in $\mathbb{R}^n$.

2. Critical points of semi-Morse functions

The equation $\Delta u = 1$ in a concentric annulus, with zero Dirichlet condition on the boundary, shows that the elements of $\mathbb{K}$ are not necessarily Morse critical points, as in that case $\mathbb{K}$ is a circle. We shall show however that critical points of any solution u to (1) are semi-Morse, in the sense that the Hessian matrix does not vanish at those points. We also term semi-Morse the functions all of whose critical points are semi-Morse. The structure of the critical set of semi-Morse functions has been already studied (see [13]).

Lemma 1. *If u is a negative solution to problem (1), then $\mathrm{L}[u] > 0$ in $\overline{\Omega}$.*

Proof. Given that the coefficients a, b and c are analytic on $\mathbb{R}^2$ and f is analytic on $\mathbb{R}$, we have, according to [9], that any solution u to (1) is analytic and can be continued as a real analytic function across the real analytic boundary $\partial\Omega$. Let $x_m \in \Omega$ be a point where u attains its minimum value. Given the ellipticity of L we should have $\mathrm{L}[u](x_m) \geq 0$. However if $\mathrm{L}[u](x_m) = 0$ it would follow that $f(u(x_m)) = 0$ and the function $u_m \equiv u(x_m)$ would be a solution of the elliptic equation

$$\mathrm{L}[v] = f(v).$$

As u is also a solution of this equation the tangency principle (see [1, Theorem 2.1.3]) implies that $u \equiv u_m$, thus leading to a contradiction. To finish we note that, as f is an increasing function, then for every $x \in \Omega$.

$$\mathrm{L}[u](x) = f(u(x)) \geq f(u(x_m)) > 0. \quad \square$$

Corollary 1. *If problem (1) possesses a negative solution, that solution is a semi-Morse function.*

We now provide a result concerning critical sets of semi-Morse functions.

Lemma 2. *Let Ω be a planar domain with smooth boundary $\partial\Omega$, and let u be a semi-Morse function defined on an open neighborhood of $\overline{\Omega}$. If all of the critical points of u belong to Ω, then the critical set of u is made up of finitely many isolated critical points, and finitely many regular analytic Jordan curves.*

Proof. The proof relies on the fact that locally the critical set of a semi-Morse function is either an isolated point or a curve (see [7,13]). The result follows now, given that Ω is bounded and there are no critical points on $\partial\Omega$. $\square$