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Positive radial solutions for a class of quasilinear boundary value problems in a ball

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ABSTRACT

We prove the existence and nonexistence of positive radial solutions for the boundary value problems

$$\begin{cases} -\Delta_p u = h(u) + \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, Ω is the open unit ball in $\mathbb{R}^n$, $h, f : (0, \infty) \to \mathbb{R}$ are allowed to be singular at 0, f is asymptotically p-linear, and λ is a positive parameter.

1. Introduction

Consider the boundary value problem

$$\begin{cases} -\Delta_p u = h(u) + \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $h, f : (0, \infty) \to \mathbb{R}$ may be singular at 0, and λ is a positive parameter. We are interested in the case when f is asymptotically p-linear in the sense that there exists $m \in (0, \infty)$ such that $\lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = m$.

The existence of positive solutions to (1.1) in the case $p = 2$, $h \equiv 0$, and $f : [0, \infty) \to \mathbb{R}$ is continuous and asymptotically linear was obtained by Ambrosetti and Hess [1] for $f(0) \geq 0$, and Ambrosetti et al. [2] for $f(0) < 0$. The results in [2,1] were obtained via topological degree and bifurcation techniques. Let us briefly recall the literature concerning related singular problems. When $p = 2$, the problem

$$\begin{cases} -\Delta u = \dfrac{K(x)}{u^\beta} + \lambda u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $\beta, q > 0$, $\lambda \geq 0$, arises in the study of nonNewtonian fluids and the theory of heat conduction in electrically conducting materials (see [3,4]). Such problems have been investigated by many authors. The case when $\lambda = 0$ and $K(x)$ is positive was studied by Crandall et al. [5], Del Pino [6], Fulks and Maybee [4], Gomes [7], and Lazer and McKenna [8]. The case when $\lambda \neq 0$ was studied in [9–13]. In [13], assuming $K(x) \equiv -1$, Zhang proved the existence of a positive solution for ($*$) when $\beta, q \in (0, 1)$, and λ is large enough. In [9], assuming $K(x) \equiv 1$, Coclite and Palmieri proved the existence of solutions for all $\lambda \geq 0$ if $q \in (0, 1)$, and, if $q \geq 1$, the existence of $\lambda^* > 0$ such that ($*$) has a

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solution for $\lambda \in [0, \lambda^*)$ and no solutions for $\lambda > \lambda^*$. Positive solutions of the general problem $(*)$ were obtained by Shi and Yao [10] for $\beta, q \in (0, 1)$, Stuart [11] for $\beta > 0, q < 1$, Sun et al. [12] for $\beta \in (0, 1)$, $1 < q < \frac{n+2}{n-2}$. Note that, except for [9] where f is allowed to be linear, the above references concerning $(*)$ dealt with the case when f is sublinear or superlinear at ∞.

Existence results for (1.1) when $p = 2, h \not\equiv 0$ and f is asymptotically linear and is allowed to be singular at 0 were established by the author in [14], which extended a result by Zhang [15]. Note that the proof in [14] depends heavily on the linearity of the Laplace operator and cannot be applied for the general p-Laplacian problem (1.1). In this paper, we study the existence of positive radial solutions for (1.1) when Ω is the unit ball and f is asymptotically p-linear. Thus, we shall consider the ODE problem

$$\begin{cases} -(r^{n-1}A(u'))' = r^{n-1}(h(u) + \lambda f(u)), & 0 < r < 1, \\ u'(0) = 0, \quad u(1) = 0, \end{cases} \tag{1.2}$$

where $A(z) = |z|^{p-2}z$. We shall prove in Theorem 2.1 below that (1.1) has a positive solution u_λ satisfying

$$u_\lambda \geq \left(\frac{\lambda \varepsilon_1}{\lambda_\infty - \lambda}\right)^{\frac{1}{p-1}} \phi_1 \quad \text{in } (0, 1),$$

when λ is sufficiently close to λ_∞ on the left, where $\lambda_\infty = \lambda_1/m$, ε_1 is a positive number, λ_1 is the first eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions and ϕ_1 is the associated positive eigenfunction with $\|\phi_1\|_\infty = 1$. In Theorem 2.2, we give conditions for (1.1) to have a positive solution for $\lambda < \lambda_\infty$, and no solution for $\lambda \geq \lambda_\infty$. In particular, our results when applied to the model case

$$\begin{cases} -\Delta_p u = \dfrac{a}{u^\beta} + \lambda \left(\dfrac{b}{u^\delta} + u^{p-1} e^{\frac{1}{u^\gamma + 1}}\right) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a ball, $a, b \in \mathbb{R}$, $\beta, \delta \in (0, 1)$, $\gamma \in (0, p-1]$, give the existence of a positive radial solution when λ is sufficiently close to λ_1 on the left, and, if $a, b > 0$, the existence of a positive radial solution if and only if $\lambda < \lambda_1$. Our approach is based on the Schauder Fixed Point Theorem. To be more precise, to prove Theorem 2.1, we look for solutions of (1.2) as fixed points of an associated compact operator in the set $\mathbf{K} = \{v \in C[0, 1] : c\phi_1 \leq v \leq M\phi_1\}$ for suitable choices of positive numbers c and M. To prove Theorem 2.2, we first show that for $\lambda < \lambda_\infty$ and suitable chosen $c, M > 0$, the modified problem

$$\begin{cases} -(r^{n-1}A(u'))' = r^{n-1}(h(\max(u, c\phi_1)) + \lambda f(\max(u, c\phi_1))), & 0 < r < 1, \\ u'(0) = 0, \quad u(1) = 0, \end{cases}$$

has a solution u which is a fixed point of a compact operator on the set $\mathbf{C} = \{v \in C[0, 1] : v \leq M\phi_1\}$, and then show that $u \geq c\phi_1$ in (0, 1). The nonexistence result when $\lambda \geq \lambda_\infty$ is proved with the aid of a strong comparison principle by Prashanth [16].

2. Existence results

We shall denote the norms in $L^p(0, 1)$, $C^1[0, 1]$, and $C^{1,\alpha}[0, 1]$ by $\|.\|_p$, $|.|_1$, and $|.|_{1,\alpha}$ respectively. We make the following assumptions.

(A1) $f, h : (0, \infty) \to \mathbb{R}$ are continuous.

(A2) There exist positive numbers ε_0, a, m such that

$$\lim_{u \to \infty} \frac{f(u)}{u^{p-1}} = m,$$

and

$$f(u) \geq mu^{p-1} + \varepsilon_0$$

for $u > a$.

(A3) There exist numbers $\alpha, \delta \in (0, 1)$ and $k_0 > 0$ such that

$$|h(u)| \leq k_0 u^{-\alpha}$$

for $u > 0$, and

$$\limsup_{u \to 0^+} u^\delta |f(u)| < \infty.$$

Note that from a result by Brock [17], ϕ_1 is radially symmetric and decreasing, and thus solves

$$\begin{cases} -(r^{n-1}A(\phi_1'))' = \lambda_1 r^{n-1} \phi_1^{p-1}, & 0 < r < 1, \\ \phi_1'(0) = 0, \quad \phi_1(1) = 0. \end{cases}$$