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Stability of solutions for variational relation problems with applications☆

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ABSTRACT

In this paper, we prove that most of problems in variational relations (in the sense of Baire category) are essential and that, for any problem in variational relations, there exists at least one essential component of its solution set. As applications, we deduce the existence of essential components of the set of Ky Fan's points based on Ky Fan's minimax inequality theorem, the existence of essential components of the set of Nash equilibrium points for general n-person non-cooperative games, the existence of essential component of the set of solutions for vector Ky Fan's minimax inequality, the existence of essential components of the set of KKM points and the existence of essential components of the set of solutions for Ky Fan's section theorem.

1. Introduction

In 1950, Fort [1] introduced the notion of essential fixed points of a continuous mapping f from a compact metric space X into itself and proved that any mapping f can be approximated closely by a mapping whose fixed points are all essential. Owing to the non-existence of essential fixed points even for the identity mapping, Kinoshita [2] then introduced the notion of essential components of the set of fixed points and proved that for any continuous mapping from the Hilbert cube into itself, there is at least one essential component of the set of its fixed points. Inspired by their work, Wu and Jiang [3] introduced the notion of essential Nash equilibrium points for finite n-person non-cooperative games (briefly, finite games) and proved that any finite game can be closely approximated by a game whose Nash equilibrium points are all essential. Later, Jiang [4] introduced the notion of essential components of the set of Nash equilibrium points and proved that for any finite game, there is at least one essential component of the set of its Nash equilibrium points. Moreover, Kohlberg and Mertens [5] argued that for finite games a satisfactory solution concept should be therefore called an essential component of the set of Nash equilibrium points and they proved that for any finite game there are finite components of the set of Nash equilibrium points, at least one of which is essential. Later, Yu and Xiang [6] introduced the notion of essential components of the set of Ky Fan's points and deduced that every infinite n-person noncooperative game with concave payoff functions has at least one essential component of the set of its equilibrium points by the method of essential solution. The method of essential solution has been widely used in various fields recently. It plays a crucial role in the study of stability of solutions including optimal solutions, Nash equilibria, fixed points, etc. (see [7–13]).

Khanh and Luc [14,15] introduced a more general model of equilibrium problems which is called a variational relation problem (in short, **VRP**). They presented several special cases of VRP and studied the existence of its solution. They have also showed the relationship between VRP and a KKM type theorem. Fan–KKM theorem [16] is one of the most powerful and useful result in nonlinear analysis. It has several equivalent formulations, several generalizations and many applications;

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see for example [17–21] and references therein. Very recently, Lin and Chuan [22] studied KKM type theorem for a finite family of multivalued maps.

We formulate the following problem of variational relations (in short, **VR**).

Let X be a non-empty compact convex subset of a Hausdorff topological vector space and $R(x, y)$ be a relation linking $x \in X$ and $y \in X$. We consider the following problem of variational relations, denoted by (VR):

$$\{\text{Find } x^* \in X \text{ such that } R(x^*, y) \text{ holds for all } y \in X\}.$$

Motivated and inspired by research works mentioned above, in this paper, we will study the stability of the solution set $\bigcap_{y\in X}\{x \in X : R(x, y) \text{ holds}\}$ with varying R in a normed vector space, where R is called a problem in variational relations. We prove that most of problems in variational relations are essential (in the sense of Baire category) and also that any problem in variational relations has at least one essential component of the set of its solutions. As applications, we deduce the existence of essential components of the set of Ky Fan's points based on Ky Fan's minimax inequality theorem, the existence of essential components of the set of Nash equilibrium points for general n-person non-cooperative games, the existence of essential component of the set of solutions for vector Ky Fan's minimax inequality, the existence of essential components of the set of KKM points and the existence of essential components of the set of solutions for Ky Fan's section theorem. We also analysis relations between our results and results in [6,8,10,13].

2. Preliminaries

First we recall some definitions [12,23].

Definition 2.1 (*[23]*). Let (X, d) and (Y, ρ) be two metric spaces, $F : X \longrightarrow 2^Y$ be a set-valued map. Then

(1) F is said to be upper semicontinuous at $x \in X$ if, for any open subset O of Y with $O \supset F(x)$, there exists an open neighborhood $U(x)$ of x such that $O \supset F(x')$ for any $x' \in U(x)$;
(2) F is said to be upper semicontinuous on X if F is upper semicontinuous on each $x \in X$;
(3) F is said to be an *usco* mapping if F is upper semicontinuous on X and $F(x)$ is compact for each $x \in X$;
(4) F is said to be lower semicontinuous at $x \in X$ if, for any open subset O of Y with $O \bigcap F(x) \neq \emptyset$, there exists an open neighborhood $U(x)$ of x such that $O \bigcap F(x') \neq \emptyset$ for any $x' \in U(x)$;
(5) F is said to be lower semicontinuous on X if F is lower semicontinuous on each $x \in X$;
(6) F is closed if $\mathrm{Graph}(F) = \{(x, y) \in X \times Y | y \in F(x)\}$ is closed.

In [12], an $y \in F(x)$ is called an essential point if, for any open neighborhood $N(y)$ of y in Y, there is a $\delta > 0$ such that $N(y) \bigcap F(x') \neq \emptyset$ for any $x' \in X$ with $d(x, x') < \delta$. The problem x is called essential if all $y \in F(x)$ are essential. For each $x \in X$, the component of a point $y \in F(x)$ is the union of all connected subsets of $F(x)$ which contain the point y. Note that all components of $F(x)$ are connected and compact. For each $x \in X$, let $e(x)$ be a nonempty closed subset of $F(x)$; $e(x)$ is called an essential set of $F(x)$ if, for each open subset O of Y with $O \supset e(x)$, there exists $\delta > 0$ such that $F(x') \bigcap O \neq \emptyset$ for each $x' \in X$ with $d(x, x') < \delta$. If a component $C(x)$ of $F(x)$ is essential, then $C(x)$ is called an essential component of $F(x)$.

Remark 2.1 (*[12]*). It is easy to see that (1) the problem $x \in X$ is essential if and only if the mapping $F : X \longrightarrow 2^Y$ is lower semicontinuous at x; (2) $e_1(x), e(x) \subset F(x)$ are nonempty closed subset and $e_1(x) \subset e_2(x)$. If $e_1(x)$ is essential, then $e_2(x)$ is also essential.

We also need the following three results which are due to Fort [1, Theorem 2], Yu et al. [12, Theorem 3.1(1) where condition (c) is unnecessary] and Yang and Yu [8, Lemma 3.3], respectively:

Lemma 2.1 (*[1]*). *If X is complete and $F : X \longrightarrow 2^Y$ is a usco map, then the set of points where F is lower semicontinuous is a dense residual set in X.*

Lemma 2.2 (*[12]*). *If $F : X \longrightarrow 2^Y$ is a usco mapping, then for each $x \in X$, there exists at least one minimal essential set of $F(x)$.*

Lemma 2.3 (*[8]*). *Let X, Y, Z be three metric spaces, $S_1 : Y \longrightarrow 2^X$ and $S_2 : Z \longrightarrow 2^X$ be two set-valued mappings. Suppose that there exists at least one essential component of $S_1(y)$ for each $y \in Y$ and there exists a continuous single-valued mapping $T : Z \longrightarrow Y$ such that $S_2(z) \supset S_1(T(z))$ for each $z \in Z$. Then, there exists at least one essential component of $S_2(z)$ for each $z \in Z$.*

Lemma 2.4 (*[23]*). *Let X and Y be two topological spaces with Y compact. If F is a closed set-valued mapping from X to Y, then F is upper semi-continuous.*

The following result is an inequality concerning the Hausdorff metric between two nonempty compact subsets in a metric space due to Yu and Zhou [11, Lemma 3.1], whose applications can be seen in the next section.