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Exponential attractors for semigroups in Banach spaces

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ABSTRACT

Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup on a Banach space X, and $\mathscr{A}$ be the global attractor for $\{S(t)\}_{t\geqslant 0}$. We assume that there exists a T^* such that $S \triangleq S(T^*)$ is of class C^1 on a bounded absorbing set $B_{\epsilon_0}(\mathscr{A})$ and $S : B_{\epsilon_0}(\mathscr{A}) \to B_{\epsilon_0}(\mathscr{A})$, and furthermore, the linearized operator L at each point of $B_{\epsilon_0}(\mathscr{A})$ can be decomposed as $L = K + C$ with K compact and $\|C\| < \lambda < 1$; then we prove the existence of an exponential attractor for the discrete semigroup $\{S^n\}_{n=1}^{\infty}$ in the Banach space X. And then we apply the standard approach of Eden et al. (1994) [9] to obtain the continuous case. Here $B_{\epsilon_0}(\mathscr{A})$ denotes the ϵ_0-neighborhood of $\mathscr{A}$ in Banach space X, and $\|C\|$ denotes the norm of the operator C.

We prove, as a simple application, the existence of an exponential attractor for some nonlinear reaction–diffusion equations with polynomial growth nonlinearity of arbitrary order.

1. Introduction

In this paper, we are mainly concerned with the existence of exponential attractors of infinite dimensional dynamical systems on general Banach space X. As we know, the global attractor and the exponential attractor are both important concepts in infinite dynamical systems, since they reveal the long time global behavior of the system. Since the 1980's, many authors have started to consider the existence and some properties of the global attractors; see [1–7], for example.

An exponential attractor, in contrast to a global attractor, enjoys a uniform exponential rate of convergence of its solutions once the solution is inside an invariant absorbing set. Because of this, exponential attractors possess a deeper and more practical property, and they remain more robust under perturbations and numerical approximations than global attractors. Hence, from 1994, many authors began to study the existence of exponential attractors; see [8–12], for example.

It follows from the process of development of the existence of exponential attractors (see [13]) that the main methods are as follows. In 1994, Eden et al. [9] gave a method firstly intended to construct exponential attractors via the squeezing property in Hilbert space. In 1995, Babin and Nicolaenko [8] extended the above result by means of the smooth property (see also [14]). In 2000, Efendiev et al. [11] proved the existence of an exponential attractor if the smooth property is replaced by the asymptotic smooth property (see also [15]). Also by proving Lipschitz continuity between a Banach space and its Hilbert subspace (see [10]), they obtained the existence of an exponential attractor in Banach space. Besides that, in 2002, Málek and Pražák [12] proved the existence of an exponential attractor by the ℓ-trajectory method. These methods have been applied to a variety of concrete equations; see [13,16–19] and references therein.

On the other hand, we note that in 2001, Dung and Nicolaenko proved the existence of an exponential attractor in Banach space; see [20]. That is:

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Theorem 1.1. *Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup on a Banach space X. Assume that the semigroup $\{S(t)\}_{t\geqslant 0}$ has a* compact absorbing *set $Y \subset X$ and there exists T^* such that $S \triangleq S(T^*) : Y \mapsto Y$ satisfies the following conditions:*

(i) *S is of class C^1 on the set Y;*
(ii) *the linearized operator L can be decomposed as $L = K + C$ with K compact and $\|C\| < \lambda < 1$.*

Furthermore, the map $(t, x) \mapsto S(t)x$ is Lipschitz on $[0, T] \times B$, $\forall T > 0$, $\forall B \subset Y$ bounded.
Then $\{S(t)\}_{t\geqslant 0}$ possesses an exponential attractor $\mathscr{M}$ on Y.

As we know that, sometimes, in the process of proof of the existence of a global attractor for a semigroup $\{S(t)\}_{t\geqslant 0}$, one can easily obtain the existence of a bounded absorbing set in a strong space, and then, applying the compact embedding theorem, one can also easily obtain a compact absorbing set in the weak space. However, if we want to get the existence of an exponential attractor in the strong space, the hypothesis of the semigroup having a compact absorbing set is difficult to uphold.

At the same time, we again refer the reader to the paper [11] where the authors established the existence of an exponential attractor in Banach space by using the asymptotic smooth property (as it was called by EMS 2000) with the assumption that the semigroup has only a bounded absorbing set. Comparing with this, it should be noted that if the semigroup is differential, then our following result is more handy than that of [11] and furthermore, our results give more concrete estimates of the dimension of the exponential attractor than [11].

In this paper, motivated by [20], we relax the hypotheses of the semigroup having a compact absorbing set in Theorem 1.1 to only needing a bounded absorbing set. (In order to compare with the results proved in [20], we organize the framework of this paper similarly to that of [20] in the following.)

Our results are formulated below:

Define S as the map induced by Poincaré sections of a Lipschitz continuous semigroup $\{S(t)\}_{t\geqslant 0}$ at the time $t = T^*$ for some $T^* > 0$; that is $S := S(T^*)$ and $S : B_{\epsilon_0}(\mathscr{A}) \to B_{\epsilon_0}(\mathscr{A})$ is a C^1 map. $\mathscr{L}(X) = \{L|L : X \to X$ bounded linear maps$\}$, $\mathscr{L}_\lambda(X) = \{L|L \in \mathscr{L}(X)$ and $L = K + C$ with K compact, $\|C\| < \lambda\}$.

For the discrete semigroup $\{S^n\}_{n=1}^{\infty}$ generated by S, our first important result is:

Theorem 1.2. *If there exists $\lambda \in (0, 1)$ such that $D_xS(x) \in \mathscr{L}_\lambda(X)$ for all $x \in B_{\epsilon_0}(\mathscr{A})$ then $\{S^n\}_{n=1}^{\infty}$ possesses an exponential attractor $\mathscr{M}_d$.*

The proof of Theorem 1.2 is based on the following slightly weaker version, whose proof will be presented in the next section.

Theorem 1.3. *If there exists $\lambda \in \left(0, \frac{1}{2}\right)$ such that $D_xS(x) \in \mathscr{L}_{\frac{\lambda}{2}}(X)$ for all $x \in B_{\epsilon_0}(\mathscr{A})$, then the discrete dynamical system $\{S^n\}_{n=1}^{\infty}$ possesses an exponential attractor $\mathscr{M}_d$.*

Furthermore, once the existence of exponential attractors for the discrete case is proved, the result for the continuous case follows in a standard manner; see [9]. We have:

Theorem 1.4. *Suppose that there is a $T^* > 0$ such that $S = S(T^*)$ satisfies the condition of Theorem 1.2 and the map $F(x, t) = S(t)x$ is Lipschitz from $[0, T] \times X$ into X for any $T > 0$. Then the flow $\{S(t)\}_{t\geqslant 0}$ admits an exponential attractor $\mathscr{M}_c$.*

As a simple application, we consider the following nonlinear reaction–diffusion equation with polynomial growth nonlinearity of arbitrary order:

$$\begin{cases} u_t - \Delta u + |u|^{p-1}u + f(x, u) = g(x) & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) = u_0 & \text{in } \Omega \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$ and $p \geqslant 2$, and the function $f \in C^2$ satisfies the following condition:

$$c_1|u|^{q-2}u - c_0 \leqslant f'(x, u) \leqslant c_2|u|^{q-2}u + c_0, \quad 1 \leqslant q < p. \tag{1.2}$$

We also assume that the initial value $u_0 \in L^2(\Omega)$ and the external force $g(x) \in L^2(\Omega)$. It is worth noticing that our case is different from that of [21] with $g(x) \in L^\infty(\Omega)$, and also different from that of [22] with some other additional assumptions.

And then combining this with Theorems 1.2 and 1.4, we prove:

Theorem 1.5. *Assume that the function $f(s)$ satisfies the assumption (1.2); then Eq. (1.1) has an exponential attractor $\mathscr{M}_c$ in $L^{2p}(\Omega)$ when the initial value $u_0 \in L^2(\Omega)$ and the external force $g(x) \in L^2(\Omega)$.*

This article is organized as follows. In Section 2, we show Theorems 1.2 and 1.4. In Section 3, we show our application result, i.e., Theorem 1.5.

Throughout this paper, $\mathscr{A}$ denotes the global attractor for semigroup $\{S(t)\}_{t\geqslant 0}$ in the Banach space X and $c, c_i,\ i = 1, 2, \ldots$, denote constants and may be different in different places.