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Pointwise well-posedness in set optimization with cone proper sets ☆

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ABSTRACT

This paper deals with the well-posedness property in the setting of set optimization problems. By using a notion of well-posed set optimization problem due to Zhang et al. (2009) [18] and a scalarization process, we characterize this property through the well-posedness, in the Tykhonov sense, of a family of scalar optimization problems and we show that certain quasiconvex set optimization problems are well-posed. Our approach is based just on a weak boundedness assumption, called cone properness, that is unavoidable to obtain a meaningful set optimization problem.

1. Introduction

Many optimization problems, where a set-valued objective map F is considered, arise in fields such as Economics, Engineering or Natural Sciences (see, for example, [1–3]).

Hence, in recent years, set-valued optimization problems have been extensively studied, mainly as a generalization of vector optimization problems.

There are two main approaches to a set-valued optimization problem, depending on the notion of minimality considered. Both of them may be considered as generalizations of the usual definition of minimality in vector optimization, because they both reduce to the usual notion of efficiency whenever F is a vector single-valued function.

The first approach to set-valued optimization, introduced in [4,5] and extensively studied in many recent works, considers the minimal boundary of the union of the images of the feasible region through the set-valued objective map F.

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An alternative approach, the so-called set optimization approach, essentially due to Kuroiwa (see [6,7]), considers an order relation among sets originally introduced independently by Young [8] and Nishnianidze [9], and studies a minimality notion induced by that order. This last approach seems to be more natural and interesting than the traditional one, whenever one needs to consider preferences over sets.

An introductory and simple example of a possible application of the approach mentioned above is quoted in [7]. This approach was also used from a theoretical point of view to obtain a minmax theorem for vector valued functions (see [10]).

Recently a new approach to set-valued optimization problems was proposed by Hamel, Heyde, Löhne, Tammer and other authors (see the monograph [11] and the references therein). This approach is also based on the order structure generated by the inclusion between the conical extension of sets. One of the specific features of this theory is the use of the lattice structure of the space of conical extensions of the subsets of the image space. An application in Finance can be found in [12].

Some aspects of the set optimization theory have already been developed by various authors (see, for example, [13,14,6, 7,15–17]). These works essentially deal with existence results, duality, scalarization and Ekeland-type variational principles. Moreover, set optimization is recently used to develop a minmax theorem for vector valued functions in [10].

More recently, in [18] the authors study the well-posedness of set optimization problems where the objective values $F(x)$ are cone-bounded sets. In particular, a pointwise Tykhonov well-posedness concept is introduced and studied, which is implicitly based on a strict minimality notion due to Ha (see [17]). In this sense, let us observe that if F is a vector single-valued function, the strict minimizers coincide with those efficient solutions where F is injective and this fact plays an important role in the study of well-posedness of vector optimization problems (see [19]). Hence, the notion of strict minimizer seems to be appropriate too, in order to deal with Tykhonov well-posedness in the setting of set optimization.

Our aim in the present work is to generalize some results of [18] on Tykhonov well-posed set optimization problems from two points of view. First, by considering problems where the objective values $F(x)$ may not be cone-bounded sets. Second, by obtaining classes of Tykhonov well-posed set optimization problems based on convexity assumptions. To prove these extensions we develop a scalarization scheme that works on cone proper sets (see [16]). This is a boundedness concept weaker than other boundedness notions widely used in partially ordered spaces, such as the cone-boundedness notion considered in [18].

The scalarization process that we use may be considered as an extension of Gerstewitz's approach [20] to set optimization problems. Two previous versions of this scalarization process were already introduced in [16,21], but they are based on stronger boundedness assumptions on the values of the objective set-valued map F.

This work is structured as follows. Section 2 collects the main concepts, mathematical tools and notations used in the paper. In particular, the set-valued optimization problem is fixed and the Gerstewitz's functional, some well-known boundedness notions and the minimizer and strict minimizer concepts are recalled.

In Section 3 we study the notion of cone proper set. To be precise, we prove that this concept is weaker than other usual boundedness notions and we obtain several characterizations.

In Section 4 we prove some properties of the Gerstewitz's scalarization process introduced by Hamel and Löhne [21] under cone properness assumptions. As a consequence we derive a characterization of minimizer and strict minimizer that extends some similar results published in [16] to set optimization problems where the objective values are not cone-bounded sets.

Section 5 concerns pointwise well-posed set optimization problems. We clarify a recent well-posedness notion introduced in [18], since it is defined through a point in the topological interior of the ordering cone and we show that it does not depend on it. Moreover, we obtain the equivalence between the well-posedness of the original set optimization problem at a given strict minimizer $\bar{x}$ and the classical Tykhonov well-posedness of the scalarized problem under the assumption that the set $F(\bar{x})$ is cone proper. This result improves a similar one of [18], which was proved by assuming that the objective map is cone-bounded-valued.

In Section 6, we use an existing notion of quasiconvexity for set-valued maps in order to individuate a class of well-posed problems. This result can be seen as a generalization to the case of a set optimization problem of a classical result in the literature: every convex problem in a locally compact metric space is well-posed (see, for example, [22,23]). This last result allows us to emphasize that, also in the set optimization framework, convexity assumptions play a crucial role to guarantee the well-posedness of a given problem. Moreover, it provides an interesting application of a quasiconvexity notion for set-valued maps introduced in [24].

2. Preliminaries

Let $(Y, \| \|)$ be a normed space. As it is usual, the topological dual space of Y is denoted by Y^*. By int A and cl A we refer to the topological interior and the closure of a set $A \subseteq Y$, respectively, and we say that A is solid if int $A \neq \emptyset$. We write $\mathbb{B}$ to refer to the open unit ball of Y. We denote the nonnegative orthant of $\mathbb{R}^p$ by $\mathbb{R}^p_+$ and $\mathbb{R}_+ := \mathbb{R}^1_+$. For every $A, A_1, A_2 \in 2^Y \setminus \{\emptyset\}$ and $\alpha \in \mathbb{R}$, we denote

$$A_1 + A_2 = \{y_1 + y_2 : y_1 \in A_1, y_2 \in A_2\}, \qquad \alpha A = \{\alpha y : y \in A\}.$$

Moreover, $\emptyset + A = A + \emptyset = \emptyset$, $\alpha\emptyset = \emptyset$ and for convenience we write $y + A$ instead of $\{y\} + A$ for all $y \in Y$.

In the whole paper, Y is assumed to be quasi ordered by the relation

$$y, z \in Y, \quad y \leq_K z \iff y - z \in -K,$$