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Controllability of nonlinear fractional dynamical systems[☆]

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ABSTRACT

In this paper we establish a set of sufficient conditions for the controllability of nonlinear fractional dynamical systems. The results are obtained by using the recently derived formula for solution representation of systems of fractional differential equations and the application of the Schauder fixed point theorem. Examples are provided to illustrate the results.

1. Introduction

During the last two decades, fractional differential equations have increasingly attracted the attention of many researchers [1–6]. Many mathematical problems in science and engineering are represented by these kinds of equations. In October 2009, Science Watch of Thomson Reuters identified this area as an Emerging Research Front and gave an award to Metzler and Klafter for their paper "*The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics*" [7]. The theory of fractional differential equations has been extensively studied by many authors [8,9]. One of the basic qualitative behaviours of a dynamical system is stability. This problem has been discussed for fractional dynamical systems in [10,11]. Besides the stability problem, another most important qualitative behaviour of a dynamical system is controllability. This means that it is possible to steer any initial state of the system to any final state in some finite time using an admissible control.

The concept of controllability plays a major role in both finite and infinite dimensional spaces, that is, systems represented by ordinary differential equations and partial differential equations. So it is natural to study this concept for dynamical systems represented by fractional differential equations. Controllability of linear systems in finite dimensional spaces is well established [12]. Many articles and monographs on control theory and its application were published during the early years (see [13]); we will mention just the books [14–19]. The problem of controllability of nonlinear systems and

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integrodifferential systems including delay systems has been studied by many researchers [20]. Controllability of fractional dynamical systems in finite dimensional space is discussed for example in [21–23,18,24]. But, as far as we know, no work has been reported on the controllability of nonlinear fractional dynamical systems in the literature. The motivation of this work is to fill this gap. Therefore, in this paper we study the controllability of nonlinear fractional dynamical systems and fractional integrodifferential systems by using the Schauder fixed point theorem.

The organization of this article is as follows. Section 2 contains some basic definitions and properties of fractional calculus and some preliminary results related to the controllability of linear fractional systems. In Section 3, we obtain the sufficient controllability conditions via one of the fixed point methods, namely, the Schauder fixed point theorem for nonlinear fractional dynamical systems. In Section 4, we establish the controllability of nonlinear fractional integrodifferential systems. Finally, Section 5 presents simple examples which illustrate the theoretical results.

2. The fractional framework and preliminaries

In this section we introduce some well known fractional operators and special functions, along with a set of properties that will be of use as we proceed in our discussion (see, for example, [25,9]).

Let α, $beta > 0$, with $n-1 < \alpha < n$, $n-1 < \beta < n$, and $n \in \mathbb{N}$, $[a, b] \subset \mathbb{R}$, D the usual differential operator, and let f be a suitable real function (for example, in general, it is sufficient to have $f \in L_1(a, b)$). Let $\mathbb{R}^m$ be the m-dimensional Euclidean space. The following definitions and properties are well known, for α, $\beta > 0$ and f as a suitable function (see, for instance, [9,25]):

(a) Riemann–Liouville fractional operators (RLFO):

(i) The left RLFO:

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,\mathrm{d}t \quad (x > a) \tag{2.1}$$

$$(D^{\alpha}_{a+}f)(x) = D^n(I^{n-\alpha}_{a+}f)(x) \quad (x > a). \tag{2.2}$$

(ii) The right RLFO:

$$(I^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,\mathrm{d}t \quad (x < b) \tag{2.3}$$

$$(D^{\alpha}_{b-}f)(x) = D^n(I^{n-\alpha}_{b-}f)(x), \quad (x < b). \tag{2.4}$$

(b) The Caputo fractional derivative:

$$({}^{C}D^{\alpha}_{a+}f)(x) = (I^{n-\alpha}_{a+}D^n f)(x) \quad (x > a), \tag{2.5}$$

and in particular $I^{\alpha}_{0+}\,{}^{C}D^{\alpha}_{0+}f(t) = f(t) - f(0)$ $(0 < \alpha < 1)$.

The following relation is well known:

$$(D^{\alpha}_{a+}f)(x) = ({}^{C}D^{\alpha}_{a+}f)(x) + \sum_{j=0}^{n-1}\frac{f^{(j)}(a)}{\Gamma(1+j-\alpha)}(x-a)^{j-\alpha}. \tag{2.6}$$

Thus, we have

$$({}^{C}D^{\alpha}_{a+}1) = 0 \tag{2.7}$$

$$(D^{\alpha}_{a+}1) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{2.8}$$

The following properties of the aforementioned operators are especially interesting:

(i) $({}^{C}D^{\alpha}_{a+}1) = 0$.

(ii) $(D^{\alpha}_{a+}1) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$.

(iii) $I^{\alpha}_{a+}(f(t)+g(t)) = I^{\alpha}_{a+}f(t) + I^{\alpha}_{a+}g(t)$.

(iv) $I^{\alpha}_{a+}I^{\beta}_{a+}f(t) = I^{\alpha+\beta}_{a+}f(t) = I^{\beta}_{a+}I^{\alpha}_{a+}f(t)$.

(v) $D^{\alpha}_{a+}I^{\alpha}_{a+}f(t) = f(t)$.

(vi) $I^{\alpha}_{a+}\,D^{\alpha}_{a+}f(t) = f(t) - \frac{\left(I^{1-\alpha}_{a+}f\right)(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}$ $(0 < \alpha < 1)$.

(vii) $I^{\alpha}_{a+}\,{}^{C}D^{\alpha}_{a+}f(t) = f(t) - f(a)$, $0 < \alpha < 1$.

(viii) In general, $D^{\alpha}_{a+}\,D^{\beta}_{a+}f(t) \neq D^{\alpha+\beta}_{a+}f(t)$ and $D^{\alpha}_{a+}D^{\beta}_{0+}f(t) \neq D^{\beta}_{a+}D^{\alpha}_{0+}f(t)$.

From the above we observe that, in general, both the Riemann–Liouville and the Caputo fractional operators possess neither semigroup nor commutative properties, which are inherent to the derivatives to integer order. However, with some restrictions, for example $0 < \alpha < 1$ and if f is a continuous function in $[a, b]$, both properties hold true for both of the aforementioned operators. There are other kinds of fractional derivatives in the literature. The Caputo fractional derivative is more often used in applied research. For basic facts and other results on fractional differential equations, one can refer to the books [9,25–27]. For brevity of notation let us take I^{α}_{0+} as I^{α} and ${}^{C}D^{\alpha}_{0+}$ as D^{α}.