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Edelstein–Suzuki-type fixed point results in metric and abstract metric spaces

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ABSTRACT

Generalizations of the Edelstein–Suzuki theorem [T. Suzuki, A new type of fixed point theorem in metric spaces, Nonlinear Anal. TMA 71 (2009), 5313–5317], including versions of the Kannan, Chatterjea and Hardy–Rogers-type fixed point results for compact metric spaces, are proved. Also, abstract metric versions of these results are obtained. Examples are presented to distinguish our results from the existing ones.

1. Introduction

In 1962, M. Edelstein proved the following version of the Banach contraction principle.

Theorem 1.1 ([1]). *Let (X, d) be a compact metric space and let $T : X \to X$ be a self-mapping. Assume that*

$$d(Tx, Ty) < d(x, y)$$

holds for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point in X.

Recently (in 2008 and 2009), T. Suzuki proved generalized versions of both Banach's and Edelstein's basic results, and thus initiated a lot of work in this direction. Several authors obtained variations and refinements of his "non-compact" result [2]. Thus, e.g., Suzuki-type versions of Kannan's and other fixed point theorems were proved in [3–8].

The "compact" (Edelstein-type) result was obtained by Suzuki in [9].

Theorem 1.2 ([9]). *Let (X, d) be a compact metric space and let $T : X \to X$. Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \Longrightarrow d(Tx, Ty) < d(x, y) \tag{1.1}$$

holds for all $x, y \in X$. Then T has a unique fixed point in X.

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This theorem did not attract that much attention (some results were given in [7,8]). We prove in this paper generalizations of this theorem, including the Kannan, Chatterjea and Hardy–Rogers-type fixed point results (see [10]), in the Edelstein–Suzuki setting.

Also, abstract (cone) metric versions of these results are obtained, where the cone metric is taken in the sense of [11–13]. We note that a cone metric version of the Edelstein theorem was given already in the initial paper of Huang and Zhang [11], but with the condition that the underlying cone is regular. A "non-compact" result of this kind was proved by Farajzadeh et al. in [14].

Examples are presented to distinguish our results from the existing ones.

2. Preliminaries

Let E be a real Banach space with θ as the zero element and let P be a subset of E with the interior $\operatorname{int} P$. The subset P is called a *cone* if: (a) P is closed, nonempty and $P \neq \{\theta\}$; (b) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ implies $ax + by \in P$; (c) $P \cap (-P) = \{\theta\}$. For the given cone P, a partial ordering $\preceq$ with respect to P is introduced in the following way: $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$, but $x \neq y$. If $y - x \in \operatorname{int} P$, we write $x \ll y$. If $\operatorname{int} P \neq \emptyset$, the cone P is called *solid*.

The cone P in E is called *normal* if there exists $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies that $\|x\| \leq K\|y\|$ (the minimal constant K satisfying the previous inequality is called the *normal constant* of P). It is known (see, e.g., [12]) that the previous condition is equivalent to the condition that there exists a norm $\|\cdot\|_1$ on E, equivalent to the given one, which is *monotone*, i.e., such that $\theta \preceq x \preceq y$ implies that $\|x\|_1 \leq \|y\|_1$.

Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies: (d_1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$; (d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$; (d_3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$. Then d is called a *cone metric* on X and (X, d) is called an *abstract (cone) metric space* [11]. For basic notions for these spaces we refer the reader to [11–13].

If the cone P is solid and normal then, using the previously mentioned equivalence, we can always suppose that the normal constant of P is $K = 1$, and that the given norm in E is monotone. In particular, we can take as a metric in X the function $D : X \times X \to \mathbb{R}$, given by $D(x, y) = \|d(x, y)\|$, since it follows from [12,15] that this metric and the cone metric d give the same topologies on X, i.e., these spaces have the same collections of open, closed, bounded and compact sets, etc. In particular, in this case we can take that the following holds for all $c, d \in E$:

$$\theta \preceq c \ll d \Longrightarrow \|c\| < \|d\| \tag{2.1}$$

(see also [14, Lemma 2.1]).

3. Results

Our first result is an Edelstein–Suzuki variant of the Hardy–Rogers fixed point theorem.

Theorem 3.1. *Let (X, d) be a compact metric space and let $T : X \to X$. Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \Longrightarrow d(Tx, Ty) < Ad(x, y) + Bd(x, Tx) + Cd(y, Ty) + Dd(x, Ty) + Ed(y, Tx) \tag{3.1}$$

holds for all $x, y \in X$, where the nonnegative constants A, B, C, D, E satisfy that

$$A + B + C + 2D = 1 \quad \text{and} \quad C \neq 1.$$

Then T has a fixed point in X. If $E \leq B + C + D$ then the fixed point of T is unique.

Proof. Define

$$\beta = \inf\{d(x, Tx) : x \in X\}.$$

There exists a sequence $\{x_n\}$ in X such that $\lim_n d(x_n, Tx_n) = \beta$. Since X is compact, we can suppose that there are $v, w \in X$ such that $\lim_n x_n = v$ and $\lim_n Tx_n = w$. Then

$$\lim_n d(x_n, w) = \lim_n d(x_n, Tx_n) = d(v, w) = \beta.$$

In order to prove that $\beta = 0$, suppose, to the contrary, that $\beta > 0$. Then there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Longrightarrow \frac{2}{3}\beta < d(x_n, w) \quad \text{and} \quad d(x_n, Tx_n) < \frac{4}{3}\beta.$$

It follows that, for $n \geq n_0$, $\frac{1}{2}d(x_n, Tx_n) < \frac{2}{3}\beta < d(x_n, w)$ and (3.1) implies that

$$d(Tx_n, Tw) < Ad(x_n, w) + Bd(x_n, Tx_n) + Cd(w, Tw) + Dd(x_n, Tw) + Ed(w, Tx_n)$$