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Periodic solutions of abstract functional differential equations with infinite delay

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ABSTRACT

We characterize the existence of periodic solutions for a class of abstract retarded functional differential equations with infinite delay. We apply our results to the nonlinear equation

$$x'(t) = Ax(t) + L(x_t) + N(x_t) + f(t), \quad t \in \mathbb{R},$$

where $A : D(A) \subset X \to X$ is a closed operator defined on a Banach space X, L is a bounded linear map, $N : \mathcal{B} \to X$ is a continuous function defined on an appropriate phase space $\mathcal{B}$ and $f \in L^p(\mathbb{T}, X)$.

1. Introduction

Motivated by the fact that abstract retarded functional differential equations (abbreviated, ARFDE) with infinite delay arise in many areas of applied mathematics, this type of equations has received much attention in recent years. In particular, the problem of the existence of periodic, almost periodic and asymptotically almost periodic solutions has been considered by several authors. We refer the reader to the books [1,2], and to the papers [3–10] and the references listed therein for information on this subject.

In this work, we are concerned with the existence of periodic solutions for a class of linear and semi-linear abstract retarded functional differential equations with infinite delay.

Let X be a Banach space endowed with a norm $\|\cdot\|$. In this paper, we study the existence of periodic solutions for the class of abstract functional differential equations described in the form

$$x'(t) = Ax(t) + L(x_t) + f(t), \quad t \in \mathbb{R}. \tag{1.1}$$

In this description $x(t) \in X$ and the history $x_t : (-\infty, 0] \to X$, given by $x_t(\theta) = x(t+\theta)$ for $\theta \le 0$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically. We will assume that $L : \mathcal{B} \to X$ is a bounded linear map and $A : D(A) \subseteq X \to X$ is a closed linear operator. Moreover, $f : \mathbb{R} \to X$ is a locally p-integrable and 2π-periodic function for $1 \le p < \infty$.

As usual, we represent by $[D(A)]$ the Banach space $D(A)$ endowed with the graph norm

$$\|x\|_{[D(A)]} = \|x\| + \|Ax\|.$$

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A similar problem for equations with finite delay has been considered in [11]. In this note, we will show that we can extend Theorem 3.4 in [11] to include the infinite delay case.

Next we recall the basic concepts necessary to obtain our results. Let Y, Z be Banach spaces. In what follows, we denote by $\mathcal{L}(Y, Z)$ the Banach space of bounded linear operators from Y into Z, and we abbreviate this notation to $\mathcal{L}(Y)$ in the case $Y = Z$. We begin with the concept of $\mathcal{R}$-boundedness (see [12, Definition 3.1]).

Definition 1.1. A family of operators $\mathcal{T} = \{T_i : i \in I\} \subseteq \mathcal{L}(Y, Z)$ is said to be $\mathcal{R}$-bounded if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each finite set $J \subseteq I$, $T_i \in \mathcal{T}$, $y_i \in Y$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables ε_i on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality

$$\left\| \sum_{i\in J} \varepsilon_i T_i y_i \right\|_{L^p(\Omega,Z)} \leq C \left\| \sum_{i\in J} \varepsilon_i y_i \right\|_{L^p(\Omega,Y)}$$

is verified. The smallest of the constant C is called $\mathcal{R}$-bound of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$.

To complete these concepts, we define the UMD spaces. But, since we just will use some results from the literature, it is enough for us to present a simple definition of the UMD space. A Banach space Z is said to be UMD if the Hilbert transform is bounded on $L^p(\mathbb{R}, Z)$ for some (and then for all) $1 < p < \infty$.

Next we denote $\mathbb{T}$ the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. We will use the identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$. Specifically, in what follows for $1 \leq p < \infty$ we denote by $L^p(\mathbb{T}, Y)$ the space of 2π-periodic p-integrable functions from $\mathbb{R}$ into Y. Similarly, the notation $W^{1,p}(\mathbb{T}, Y)$ stands for the Sobolev space of 2π-periodic functions $f : \mathbb{R} \to Y$ such that $f' \in L^p(\mathbb{T}, Y)$. Moreover, q will be used to denote the conjugate exponent of p.

2. Existence of periodic solutions of ARFDE with finite delay

In this section, we study the existence of periodic solutions of a linear ARFDE with finite delay 2π. Specifically, we will be concerned with the equation

$$x'(t) = Ax(t) + F(x_t) + f(t), \quad t \in \mathbb{R}, \tag{2.1}$$

where the function $x_t : [-2\pi, 0] \to X$ is given by $x_t(\theta) = x(t+\theta)$ for $-2\pi \leq \theta \leq 0$, $A : D(A) \subseteq X \to X$ is a closed linear operator and $f : \mathbb{R} \to X$ is a locally p-integrable and 2π-periodic function, for $1 < p < \infty$.

The existence of periodic solutions for this equation when $F : L^p([-2\pi, 0], X) \to X$ is a bounded linear map was studied in [11]. In this paper, we use the following concept of solution.

Definition 2.1. Let $1 \leq p < \infty$ and let $f : \mathbb{R} \to X$ be a locally p-integrable function. We say that $x : \mathbb{R} \to X$ is a strong L^p-solution of Eq. (2.1) if $x(\cdot) \in C(\mathbb{R}, [D(A)]) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}, X)$ and (2.1) holds a.e. for $t \in \mathbb{R}$.

Eq. (2.1) is usually studied on the space $C([-2\pi, 0], X)$. For this reason, we assume that F is a bounded linear map from $C([-2\pi, 0], X)$ into X that can be extended to a bounded linear map $\widehat{F} : L^p([-2\pi, 0], X) \to X$. Hence, we can consider the equation

$$x'(t) = Ax(t) + \widehat{F}(x_t) + f(t), \quad t \in \mathbb{R}. \tag{2.2}$$

It is clear that a 2π-periodic solution of (2.2) is also a 2π-periodic solution of (2.1). Moreover, if $e_k(\theta) = e^{ik\theta}$ for $-2\pi \leq \theta \leq 0$ and $k \in \mathbb{Z}$, then

$$B_k x = \widehat{F}(e_k x) = F(e_k x).$$

We denote $\sigma_{\mathbb{Z}}(\Delta) = \{k \in \mathbb{Z} : ikI - A - B_k \text{ has no inverse}\}$. We have the following result.

Theorem 2.1. *Let X be a UMD space and $1 < p < \infty$. Assume that F can be extended as a bounded linear map on $L^p([-2\pi, 0], X)$. Then the following conditions are equivalent.*

(a) *For every function $f \in L^p(\mathbb{T}, X)$ there exists a unique locally p-integrable and 2π-periodic strong solution of (2.1).*
(b) *$\sigma_{\mathbb{Z}}(\Delta) = \phi$ and $\{ik(ikI - A - B_k)^{-1} : k \in \mathbb{Z}\}$ is R-bounded.*

Proof. Assume that for every function $f \in L^p(\mathbb{T}, X)$, there exists a unique locally p-integrable and 2π-periodic strong solution of (2.1), say $x(\cdot)$. Then, by the assumption on F, we have that $x(\cdot)$ is also the unique locally p-integrable and 2π-periodic strong solution of (2.2). Hence, (b) follows from [11, Theorem 3.4]. Conversely, assume that $\sigma_{\mathbb{Z}}(\Delta) = \phi$ and $\{ik(ikI - A - B_k)^{-1} : k \in \mathbb{Z}\}$ is R-bounded. From [11, Theorem 3.4], we obtain that for every function $f \in L^p(\mathbb{T}, X)$ there exists a unique locally p-integrable and 2π-periodic strong solution $x(\cdot)$ of (2.2). Since $\hat{F}$ is an extension of F, we conclude that $x(\cdot)$ is the unique solution of (2.1), proving (a). □