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# Subharmonic solutions with prescribed minimal period for some second-order impulsive differential equations[☆]

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ABSTRACT

This paper uses critical point theory and variational methods to investigate the subharmonic solutions with prescribed minimal period for a class of second-order impulsive differential equations. The conditions for the existence of subharmonic solutions are established. Our results rectify some known results in the literature and will allow us, in the future, to deal with the subharmonic solutions for more extensive impulsive problems.


## 1. Introduction

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years. Some classical tools such as the coincidence degree theory of Mawhin, fixed point theorems in cones, the method of lower and upper solutions, and the critical point theory have been widely used to get the solutions of impulsive differential equations. For the theory and classical results see the monographs [1–4]. Some recent development and applications of impulsive differential equations can be seen in [5–12]. We point out that in a second-order differential equation $u'' = f(t, u, u')$, one usually considers impulses in the position $u$ and the velocity $u'$. However, in the motion of spacecraft one has to consider instantaneous impulses depending on the position, that result in jump discontinuities in velocity, but with no change in position [13]. The impulses only in the velocity occur also in impulsive mechanics [14].

Recently, some researchers studied the minimal period problem or homoclinic solution for some classes of Hamiltonian systems and classical pendulum equations [15–21]. The existence and nonexistence of subharmonic solutions are considered in those papers. In particular, in [15], using the variational methods and decomposition technique, Yu got some new sufficient conditions for the existence of periodic solutions with minimal period $pT$ for the following nonautonomous Hamiltonian systems:

$$x'' + F'_x(t, x) = 0.$$

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In order to prove the main result (Theorem 2.1 in [15]), one sets

$$q = \frac{2\left(A - \left(\frac{\omega}{p}\right)^2\right)}{3\beta\|x_1\|_{L^2(0,T)}}. \tag{1.1}$$

Since $q$ is a factor of integer $p$, and $q > 1$, (1.1) may not hold. Hence, the proof of Theorem 2.1 in [15] may not be complete. We give another way, in our main result Theorem 3.1, to solve the problem similarly. This method could also have been used in [15].

To the best of our knowledge, there are no papers studying the minimal period problem for impulsive differential equations. Because of the impulsive term, we cannot use the decomposition technique which is extensively used in the non-impulsive cases. So the minimal period problem for impulsive differential equations becomes more complicated to study. Nevertheless, for some special impulsive conditions we can work the problem out. This will allow us, in the future, to deal with the subharmonic solutions for more extensive impulsive conditions.

In this paper, we consider the following second-order impulsive differential equations:

$$\begin{cases} u''(t) + f(t, u(t)) = 0, & \text{a.e. } t \in J', \\ \Delta u'(t_k) = I_k(u(t_k)), & k \in Z_0, \end{cases} \tag{1.2}$$

where $f \in C(R \times R, R)$, $Z_0 = Z^+ \cup Z^-$, $J' = R \setminus \{t_k | k \in Z_0\}$, $I_k \in C(R, R^+ \cup \{0\})$, $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $u'(t_k^\pm) = \lim_{t \to t_k^\pm} u'(t)$, $0 < t_1 < \cdots < t_m < T$, $I_{k+m} = I_k$, $T \in R^+$, and $t_k = t_{m+k} - T$ if $k \in Z^+$, while $t_k = t_{m+k+1} - T$ if $k \in Z^-$.

For any integer $p \geq 2$, the norm in $L^2([0, pT], R)$ is denoted by $\|\cdot\|_0$. Consider the functional

$$\varphi(u) = \int_0^{pT} \left[\frac{1}{2}|u'(t)|^2 - G(t, u)\right] dt + \sum_{k \in K} \int_0^{u(t_k)} I_k(t) dt,$$

where $G(t, u) = \int_0^u f(t, \xi) d\xi$, $K = \{k \in Z_0 | t_k \in (0, pT)\}$, defined on the Sobolev space

$$X = \{u \in H^1([0, pT], R) | u(0) = u(pT)\},$$
$$\|u\| = \|u\|_0 + \|u'\|_0.$$

Since each $u \in X$ can be extended periodically to the whole line, we may not distinguish $u$ and its extension.

As in [11], we need to introduce a different concept of solution. Suppose that $u \in C[0, pT]$ is such that it satisfies the condition $u(0) = u(pT)$. Moreover assume that for every $k = 1, 2, \ldots, m$, $u_k = u|_{(t_k, t_{k+1})}$ is such that $u_k \in H^2(t_k, t_{k+1})$. We say that $u$ is a classical $pT$-periodic solution of (1.2) if it satisfies the equations in (1.2).

It is easy to see that $\varphi \in C^1(X, R)$. Moreover, the critical points of $\varphi$ correspond to classical $pT$-periodic solutions of (1.2). Actually, if $u \in X$ is a critical point of $\varphi$, then we have

$$\begin{aligned} \varphi'(u)v &= \int_0^{pT} u'(t)v'(t)dt - \int_0^{pT} f(t, u(t))v(t)dt + \sum_{k \in K} I_k(u(t_k))v(t_k) \\ &= -\int_0^{pT} u''(t)v(t)dt - \int_0^{pT} f(t, u(t))v(t)dt = 0 \quad \forall v \in X. \end{aligned}$$

For $k \in \{1, 2, \ldots, m\}$, choose $v \in X$ with $v(t) = 0$ for every $t \in [0, t_k] \cup [t_{k+1}, pT]$. Then

$$\int_{t_k}^{t_{k+1}} u''(t)v(t)dt + \int_{t_k}^{t_{k+1}} f(t, u(t))v(t)dt = 0,$$

which implies $u''(t) + f(t, u(t)) = 0$ a.e. on $(t_k, t_{k+1})$. Hence $u_k \in H^2(t_k, t_{k+1})$ and $u''(t) + f(t, u(t)) = 0$ a.e. on $[0, pT]$. Now, multiplying by $v \in X$ and integrating between 0 and $pT$, we get

$$\sum_{k=1}^m \Delta u'(t_k)v(t_k) = \sum_{k=1}^m I_k(u(t_k))v(t_k).$$

Hence, $\Delta u'(t_k) = I_k(u(t_k))$ for every $k = 1, 2, \ldots, m$.

However, one should caution that the solutions' minimal periods may not be $pT$. Define $\omega = \frac{2\pi}{T}$, and $p_s$ as the smallest prime factor of $p$.

The rest of the paper is organized as follows: In Section 2, we give several assumptions and important lemmas. The main theorems are formulated and proved in Section 3. And in Section 4, we give an example to demonstrate the application of our results.