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Averaging principle and hyperbolic evolution equations

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ABSTRACT

An averaging principle is derived for the abstract nonlinear evolution equation where the almost periodic right hand-side is a continuous perturbation of the time-dependent family of linear operators determining a linear evolution system. It generalizes classical Henry's results for perturbations of sectorial operators on fractional spaces. It is also proved that the main hypothesis of the nonlinear averaging principle is satisfied for general hyperbolic evolution equations introduced by Kato.

1. Introduction

We are concerned with the limit behavior with regard to $\lambda \to 0^+$ of evolution systems of the form

$$\dot{u}(t) = A(t/\lambda)u(t) + F(t/\lambda, u(t)), \quad t > 0, \qquad (P_\lambda)$$

where $\{A(t)\}_{t\geq 0}$ is a family of operators generating C_0 semigroups of bounded linear operators on a Banach space E, $F : [0, +\infty) \times E \to E$ is a continuous map satisfying the local Lipschitz condition with respect to the second variable and $\lambda > 0$ is a parameter. The so-called *averaging principle* is a well known tool in the theory of ordinary differential equations, i.e., when E is finite dimensional. Roughly speaking, it says that if F is periodic in time, then trajectories of $\dot{u}(t) = F(t/\lambda, u(t))$ converge to trajectories of the averaged equation as $\lambda \to 0^+$ (see [1]). This averaging idea is of importance when studying qualitative behavior of nonautonomous equations. It enables to perceive the dynamics of a nonautonomous equation in terms of the related averaged one. For instance, by this approach, one may examine global attractors for dissipative equations, periodic solutions and other dynamic features such as bounded or recurrent solutions. Therefore extending the method to the infinite dimension and applying it to partial differential equations is a natural and vital issue attracting much attention. The averaging principle in the infinite dimensional case was obtained by Henry [2] who assumed that the (independent of time) operator A is a sectorial one on a Banach space E and $F : [0, +\infty) \times E^\alpha \to E$, where E^α, $0 \leq \alpha < 1$, is the fractional power space determined by A, is bounded and continuous. Averaging for time dependent (set-valued) perturbations of a C_0 group generator was considered by Kamenskii et al. in [3], where A was a C_0 semigroup generator and F was an upper semicontinuous k-set contraction with respect to a measure of noncompactness. Averaging principle, in the context of attractors and the Conley–Rybakowski index, for parabolic partial differential equations on $\mathbb{R}^N$ was used by Antocci and Prizzi [4] and Prizzi [5]. Recently a version of the averaging principle has been obtained by the author in [6] where A was a C_0 semigroup generator and F a time periodic continuous perturbation.

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In this paper we look for a general averaging scheme in the abstract operator setting with time dependent A and apply it to hyperbolic evolution equations. We shall prove a general principle, a version of which can be stated as follows (cf. Theorem 2.2 and Remark 2.10).

Theorem 1.1. *Let $\{R^{(\lambda)}(t,s)\}_{t\geq s\geq 0}$, $\lambda > 0$, be linear evolution systems on a separable Banach space E, corresponding to the problems*

$$\begin{cases} \dot{u}(t) = A(t/\lambda)u(t), & t > s, \\ u(s) = \bar{u} \in E. \end{cases}$$

Suppose that

(A1) *there are $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|R^{(\lambda)}(t,s)\| \leq Me^{\omega(t-s)}$ if $t \geq s \geq 0$;*
(A2) *there exists a C_0 semigroup $\{\widehat{S}(t)\}_{t\geq 0}$ of bounded linear operators on E with the infinitesimal generator $\widehat{A}$ such that, for any $\bar{u} \in E$ and $t, s \geq 0$ with $t \geq 0$,*

$$\lim_{\lambda\to 0^+,\ \bar{v}\to\bar{u}} R^{(\lambda)}(t,s)\bar{v} = \widehat{S}(t-s)\bar{u}$$

uniformly with respect to t, s from bounded intervals;
(A3) *a continuous mapping $F : [0,+\infty) \times E \to E$ is Lipschitz on bounded subsets and has sublinear growth uniformly with respect to the second variable;*
(A4) *for each $\bar{u} \in E$, the set $\{F(t,\bar{u}) \mid t \geq 0\}$ is relatively compact and there is a locally Lipschitz mapping $\widehat{F} : E \to E$ such that, for any $\bar{u} \in E$ and $h > 0$,*

$$\widehat{F}(\bar{u}) = \lim_{T\to+\infty,\bar{v}\to\bar{u}} \frac{1}{T}\int_0^T F(\tau+h,\bar{v})\, \mathrm{d}\tau$$

uniformly with respect to h.

Then, for any (λ_n) in $(0,+\infty)$ and $(\bar{u}_n)$ in E such that $\lambda_n \to 0^+$ and $\bar{u}_n \to \bar{u}_0$ for some $\bar{u}_0 \in E$, the mild solutions $u_n : [0,+\infty) \to E$ of (P_{λ_n}) satisfying $u_n(0) = \bar{u}_n$, $n \geq 1$, converge uniformly on bounded intervals to the mild solution of the averaged problem

$$\begin{cases} \dot{u}(t) = \widehat{A}u(t) + \widehat{F}(u(t)), & t > 0, \\ u(0) = \bar{u}_0. \end{cases}$$

Assumptions (A1) and (A2) actually state that the averaging principle holds for the linear equation. Obviously, it is always the case if A is independent of time and is an infinitesimal generator of a C_0 semigroup. We will verify (A1) and (A2) for hyperbolic type linear evolution systems introduced by Kato—see Theorem 3.3. Assumption (A3) and the separability of E are to assure the existence of unique mild solutions for initial value problems associated with (P_λ), the boundedness of solutions starting from bounded sets and the relative compactness of semiorbits of relatively compact sets (see $(\mathcal{H}_1)$–$(\mathcal{H}_3)$). Finally, (A4) simply says that F has the average $\widehat{F}$. It is worth mentioning that (A4) is fulfilled if F is almost periodic with respect to time (see [7]) and it is always the case when F is time-periodic. The obtained theorem generalizes those known in the literature—see Remark 2.6.

The paper is organized as follows. Section 2 is devoted to the general version of averaging principle while in Section 3 we are concerned with its verification for abstract linear hyperbolic evolution systems. Section 4 provides an example of application to first order hyperbolic partial differential equations.

Notation

By $\mathbb{R}$ we denote the field of real numbers; by $[x]$ we mean the integer (or floor) part of $x \in \mathbb{R}$.

If X is a metric space and $B \subset X$, then ∂B and clB stand for the boundary of B and the closure of B, respectively. If $x_0 \in X$ and $r > 0$, then $B(x_0, r) := \{x \in M \mid d(x, x_0) < r\}$.

If E is a normed space, then by $\|\cdot\|$ we denote its norm. If V is another normed space then $\mathcal{L}(V, E)$ stands for the space of all bounded linear operators with domain V and values in E with the operator norm denoted by $\|\cdot\|_{\mathcal{L}(V,E)}$ or simply $\|\cdot\|$ if no confusion may appear.

2. General averaging principle

Recall that a family of bounded linear operators $\{R(t,s) : E \to E\}_{t\geq s\geq 0}$ on a Banach space E is an *evolution system* if and only if $R(t,t) = I$, $R(t,s)R(s,r) = R(t,r)$, whenever $t \geq s \geq r \geq 0$, and the mapping $(s,t) \mapsto R(t,s)\bar{u}$ is continuous for any $\bar{u} \in E$. In this section we deal with general evolution systems, i.e. we do not indicate how they are generated.

Evolution systems come up naturally in equations involving time-dependent families of linear operators. Namely, if $\{A(t)\}_{t\geq 0}$ is a family of linear operators in a Banach space E satisfying suitable assumptions, then for any $s \geq 0$ and $\bar{u} \in E$, the problem

$$\begin{cases} \dot{u}(t) = A(t)u(t), & t > s, \\ u(s) = \bar{u}, \end{cases}$$