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Convex regularization of local volatility models from option prices: Convergence analysis and rates

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ABSTRACT

We study a convex regularization of the local volatility surface identification problem for the Black–Scholes partial differential equation from prices of European call options. This is a highly nonlinear ill-posed problem which in practice is subject to different noise levels associated to bid–ask spreads and sampling errors. We analyze, in appropriate function spaces, different properties of the parameter-to-solution map that assigns to a given volatility surface the corresponding option prices. Using such properties, we show stability and convergence of the regularized solutions in terms of the Bregman distance with respect to a class of convex regularization functionals when the noise level goes to zero.

We improve convergence rates available in the literature for the volatility identification problem. Furthermore, in the present context, we relate convex regularization with the notion of exponential families in Statistics. Finally, we connect convex regularization functionals with convex risk measures through Fenchel conjugation. We do this by showing that if the source condition for the regularization functional is satisfied, then convex risk measures can be constructed.

1. Introduction

In financial markets a number of contracts are negotiated in such a way that their values are derived from other underlying assets or equities. Such derivative contracts play a fundamental role in risk management and corporate strategies. Their presence became so widespread that currently, the volume of many derivative markets surpasses the value of the corresponding underlying markets.

The development of mathematical methods for pricing derivatives has been a major reason for the expansion of derivative markets. Such theoretical achievement was recognized by the Nobel prize in Economics award to Merton and Scholes. The corresponding methods involve the solution of the Black–Scholes partial differential equation, which in turn depends on the risk-free interest rate prevalent in the market, the dividend rate, and the volatility of the underlying asset. There are many models to describe the volatility. Among those, one that is very popular with practitioners is to assume that such volatilities are functions of the form $\sigma = \sigma(t, S)$, where t is the time and S is the asset price. It is usually referred to as Dupire's local volatility model [1] and σ is called the volatility surface.

This paper is concerned with theoretical aspects of the practical problem of determining the volatility from market observed prices of European call options. This is a nonlinear ill-posed problem whose solution calls for regularization

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techniques. We propose Tikhonov regularization by means of a convex regularizing functional as an extension to the quadratic regularization that has been used previously in the inverse problem literature for this specific problem [2–4].

We address the regularization problem from the perspective of convex analysis methods and Bregman distances. On the theoretical side, our result is that this yields better convergence rates and allows for convergence in spaces different from those in the quadratic regularization setting. In fact, in some cases, the convergence of certain convex regularization expressions implies convergence in the L^1-norm. Besides those results, our approach connects with central topics in different areas of current research. Such topics include *exponential families* of probability distributions, which is an important subject in Statistics and *convex risk measures* in Risk Management and Quantitative Finance [5,6].

The connection between Bregman distances and exponential families is well established in some context [7,8], albeit in the present context our motivation in Section 5 is heuristic. From the financial intuition, it can be understood as follows: each volatility surface leads to a corresponding risk neutral measure whose expectation of the payoff are the observed derivative prices. Thus, if we are given the problem of inferring the volatility surface from market observed option prices, the use of Bregman distances leads to the choice of certain exponential families of probability distributions. The latter, can be thought of as optimal (in an appropriate sense) *a posteriori* distributions for the class of models under consideration. Indeed, under some circumstances, exponential families are connected to minimal entropy measures. This hints to yet another connection with the now classical work developed by Avellaneda et al. See [9] and references therein.

The passage of the regularized volatility to the market probability measures allows us to also connect the results to convex risk measures. In fact, in Section 6, we exhibit procedures to produce such risk measures which depend on the regularization functional. This in turn relates to Malliavin calculus results and the determination of the so-called Greeks of option prices [10].

The setting and the inverse problem: We consider a complete financial market, where cash can be borrowed at a constant *interest rate* r, and a risky stock of value $S = S(t)$ that yields a continuously compounded dividend at a constant rate q, satisfying the diffusion price process

$$dS(t) = S(t)(\nu(t, S(t))dt + \sigma(t, S(t))dW(t)), \quad t > 0, \qquad S(0) = S_0, \tag{1}$$

where $W(t)$ denotes the standard Wiener process [11]. The parameters ν and σ are called drift rate and the volatility of the underlying asset, respectively.

A *European call option* with maturity date T and strike K, on the underlying asset S, consists of the right, but not the obligation, to buy, at a price K, a unit of S at time T. In the context of complete and arbitrage-free markets, the theoretical fair price, for the European call on S, has the probabilistic representation

$$U(0, S_0; T, K, r, q, \sigma^2) = \exp(-rT)\mathbb{E}_{\mathbb{Q}}^{0,S_0}(S(T) - K)^+, \tag{2}$$

where $\mathbb{E}_{\mathbb{Q}}^{0,S_0}$ is the expected value with respect to the *risk-neutral* probability measure $\mathbb{Q}$ given that, at $t = 0$, we have $S(0) = S_0$. Here, as usual, we define

$$(S - K)^+ := \max\{S - K, 0\}.$$

The interpretation of Eq. (2) is that for each realization ω of the market, the payoff $(S(T, \omega) - K)^+$ should be brought to its present value $e^{-rT}(S(T, \omega) - K)^+$ by means of discounting by the interest rate r. Then, we average over all the possible realizations with respect to the risk neutral measure $\mathbb{Q}$. The risk neutral measure differs from the so-called subjective one in the sense that it is the one for which the discounted process $S(t)/e^{rt}$ is a martingale. For more details see [12].

In this framework the fair price for an European call option is given by the solution to the Black–Scholes equation [13]

$$U_t + \frac{1}{2}\sigma^2(t, S)S^2U_{SS} + (r - q)SU_S - rU = 0, \quad t < T, \tag{3}$$

with final condition

$$U(t = T, S) = (S - K)^+. \tag{4}$$

An important consequence of the Black–Scholes–Merton theory is that the drift rate ν in Eq. (1) does not enter into (3). Indeed, this is at the root of the concept of the risk-neutral measure $\mathbb{Q}$.

In the case where σ is a deterministic function of time only, explicit formulas for the price U are well known. See the seminal paper [13]. In this context, a careful analysis of the theoretical volatility calibration problem was carried out in [14,15].

We note that the option price U depends also on the maturity T and strike K. It satisfies the, by now classical, *Dupire* forward equation [1]

$$-U_T + \frac{1}{2}\sigma^2(T, K)K^2U_{KK} - (r - q)KU_K - qU = 0, \quad T > 0, \tag{5}$$

with the initial value

$$U(T = 0, K) = (S_0 - K)^+, \quad \text{for } K > 0. \tag{6}$$