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# Measure theory and pseudo almost automorphic functions: New developments and applications

Joël Blot [a], Philippe Cieutat [b], Khalil Ezzinbi [c,*]

[a] Université Paris 1 Panthéon-Sorbonne, Laboratoire SAMM EA 4543, 75013 Paris, France
[b] Université Versailles-Saint-Quentin-en-Yvelines, Laboratoire de Mathématiques de Versailles, UMR-CNRS 8100, 78035 Versailles cedex, France
[c] Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, B.P.2390, Marrakech, Morocco

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ABSTRACT

In this work, we establish a new concept of weighted pseudo almost automorphic functions using the measure theory. We present new results on weighted ergodic functions like completeness and composition theorems. The theory of this work generalizes the classical results on weighted pseudo almost periodic and automorphic functions. For illustration, we provide some applications for evolution equations which include reaction–diffusion systems and partial functional differential equations.


## 1. Introduction

The aim of this work is to present a new approach dealing with weighted pseudo almost automorphic functions. The concept of almost automorphy was first introduced in the literature by Bochner [1] in the earlier sixties, it is a natural generalization of the almost periodicity, for more details about this topic we refer the reader to the recent book [2] where the author gave an important overview about the theory of almost automorphic functions and their applications to differential equations. In the last decade, many authors have produced extensive literature on the theory of almost automorphy and its applications to differential equations, more details can be found in [3–14] and the references therein.

Here we use the measure theory to define an ergodic function and we investigate many interesting properties of such functions. Weighted pseudo almost automorphic functions have been studied recently and have become an interesting field

* Corresponding author. Tel.: +212 524 43 46 49; fax: +212 524 43 74 09.
*E-mail addresses:* Joel.Blot@univ-paris1.fr (J. Blot), Philippe.Cieutat@math.uvsq.fr (P. Cieutat), ezzinbi@ucam.ac.ma (K. Ezzinbi).

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in dynamical systems [15–19]. The notion of weighted pseudo almost automorphy has been introduced for the first time by Blot et al. in [15].

Unfortunately, many results developed in the literature are false and not correct, for example uniqueness of the decomposition is not unique, consequently the proof of the completeness based on the uniqueness is not correct. The aim here is to correct many results obtained in the literature and to present new more general results based on the measure theory. Many counter-examples are provided through this work to show that the theory of weighted pseudo almost automorphy provided until now is false.

Weighted pseudo-almost automorphic functions are more general than weighted pseudo-almost periodic functions which were introduced in [20,21]. In [15], the authors studied basic properties of weighted pseudo-almost automorphic functions and then used these results to study the existence and uniqueness of weighted pseudo-almost automorphic mild solutions for some abstract differential equations. Let us explain the meaning of this notion. We say that a continuous function $f$ is $\rho$-weighted pseudo almost automorphic if

$$f = g + \phi,$$

where $g$ is almost automorphic and $\phi$ is ergodic with respect to some weighted function $\rho$ in the sense that

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|\phi(t)\| \rho(t)\, dt = 0,$$

where $m(r,\rho) = \int_{-r}^{r} \rho(t)\, dt$, where $\rho$ is assumed to be positive and locally integrable. When the component $g$ is almost periodic, then $f$ is called weighted almost periodic.

Here we give a positive measure $\mu$ on $\mathbb{R}$, and we define a new concept of $\mu$-pseudo almost automorphic functions. We say that a function $f$ is $\mu$-pseudo almost automorphic if

$$f = g + \phi,$$

where $g$ is almost automorphic and $\phi$ is $\mu$-ergodic in the sense that

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|\phi(t)\| d\mu(t) = 0,$$

where $\mu([-r,r])$ is the measure of the set $[-r,r]$. One can observe that a $\rho$-weighted pseudo almost automorphic function is $\mu$-pseudo almost automorphic, where the measure $\mu$ is absolutely continuous with respect to the Lebesgue measure and its Radon–Nikodym derivative is $\rho$:

$$\frac{d\mu(t)}{dt} = \rho(t).$$

Now, the classical theory of weighted pseudo almost automorphy becomes a particular case of our approach. In this work, we investigate many important results on the new theory of $\mu$-pseudo almost automorphic functions, we study the completeness and the composition theorem on the functional space of $\mu$-pseudo almost automorphic functions. Those results have an important impact on the theory of systems. For ordinary differential equations or evolution equations, we will show that if the input function is $\mu$-pseudo almost automorphic then the system has a $\mu$-pseudo almost automorphic solution, some applications will be provided for evolution equations and partial functional differential equations.

The organization of this work is as follows: in Section 2, we give the new concept of $\mu$-pseudo almost automorphy using the measure theory. In Section 3, we give sufficient conditions for the translation invariance of the space of $\mu$-pseudo almost automorphic functions and then we state a result about the convolution product in the space of $\mu$-pseudo almost automorphic functions. In Section 4, we study the completeness of the space of $\mu$-pseudo almost automorphic functions. In Section 5, we state a composition theorem which plays a crucial role to study the existence of $\mu$-pseudo almost automorphic solutions for a perturbed system. In Section 6, we propose some applications for general evolution equations and reaction–diffusion systems. In Section 7, we propose another application for partial functional differential equations in Banach spaces, in fact we use the theory of semigroups and convolution product theorems in the space of $\mu$-pseudo almost automorphic functions to prove the existence of $\mu$-pseudo almost automorphic solutions.

## 2. $\mu$-pseudo almost automorphic functions

In this section, we define new notion of the $\mu$-ergodic functions and the $\mu$-pseudo almost automorphic functions under the light of measure theory, then we give some fundamental properties of these functions that we use in differential equations and dynamical systems. Recall that the notion of $\mu$-pseudo almost automorphy is a generalization of the pseudo almost automorphy. These functions were introduced recently in [9,10].

Throughout this work, $X$ is a Banach space and $BC(\mathbb{R}, X)$ denotes the Banach space of bounded continuous functions from $\mathbb{R}$ to $X$, equipped with the supremum norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|.$$

We denote by $\mathcal{B}$ the Lebesgue $\sigma$-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures $\mu$ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$, for all $a, b \in \mathbb{R}$ $(a \le b)$.