Nonlinear Analysis 75 (2012) 2520–2533

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Existence and multiplicity results for quasilinear elliptic exterior problems with nonlinear boundary conditions

Leiga Zhao [a,*], Anran Li [b], Jiabao Su [b]

[a] Department of Mathematics, Beijing University of Chemical Technology, Beijing 100029, People's Republic of China

[b] School of Mathematical Sciences, Capital Normal University, Beijing 100037, People's Republic of China

ARTICLE INFO

Article history:
Received 30 May 2011
Accepted 28 October 2011
Communicated by S. Carl

MSC:
35J60
58E05

Keywords:
Quasilinear elliptic equation
Exterior domain
Nonlinear boundary conditions

ABSTRACT

In this paper, we study a class of quasilinear elliptic exterior problems with nonlinear boundary conditions. Existence of ground states and multiplicity results are obtained via variational methods.

1. Introduction

In this paper, we are concerned with the following quasilinear elliptic equations with a mixed nonlinear boundary condition

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) + |u|^{q-2}u = \lambda g(x)|u|^{r-2}u, & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\partial_\nu u + b(x)|u|^{p-2}u = 0, & \text{on } \partial\Omega. \end{cases} \tag{P}$$

It is assumed that Ω is a smooth exterior domain in $\mathbb{R}^N$ ($N \geqslant 3$), that is, Ω is the complement of a bounded domain with $C^{1,\delta}$ ($0 < \delta < 1$) boundary $\partial\Omega$, ν is the unit outer normal to $\partial\Omega$, $1 < p < N$, and $\lambda > 0$ is a real parameter.

In recent years, existence, non-existence and multiplicity of solutions for problems involving the p-Laplace operator were studied by several authors, in bounded domain, see e.g. [1–5], while in unbounded domains or the entire $\mathbb{R}^N$, see e.g. [6–11] and the references therein. Problems of this type appear in many nonlinear diffusion problems, such as the mathematical model of non-Newtonian fluid [12], the chemotactic aggregation model introduced by Keller and Segel [13]. As pointed out in [14,15], certain stationary waves in nonlinear Klein–Gordon or Schrodinger equations can be reduced to this form. For some physical motivation of such nonlinear boundary conditions, see [16]. From the point of view of mathematics, similar problems with nonlinear boundary condition appear in the study of the optimal constants for the Sobolev trace embeddings. See e.g. [1,2,8,5].

* Corresponding author.
E-mail address: zhaolg@amss.ac.cn (L. Zhao).

0362-546X/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2011.10.046

In this paper, the weight functions a, b and g satisfy the following assumptions:

(a) $0 < a_0 \leqslant a(x) \in L^\infty(\Omega) \cap C^{0,\delta}(\overline{\Omega})$;
(b) $0 < b(x) \in C(\partial\Omega)$;
(g) $0 \leqslant g(x) \in L^\infty(\Omega) \cap L^{p_0}(\Omega)$ with $p_0 = p^*/(p^* - r)$ where $1 < r < p^*$, $p^* = \frac{Np}{N-p}$, and g is positive on a nonempty open subset of Ω.

It is clear that problem (P) possesses a trivial solution $u = 0$ for any $\lambda \in \mathbb{R}$. We are interested in the existence and multiplicity of weak solutions of (P).

By a weak nontrivial solution of (P), one means a nontrivial function $u \in X := E \cap L^q(\Omega)$ verifying for all $\phi \in X$ the identity

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla\phi\, dx + \int_{\partial\Omega} b(x)|u|^{p-2}u\phi\, d\sigma + \int_\Omega |u|^{q-2}u\phi\, dx = \lambda \int_\Omega g(x)|u|^{r-2}u\phi\, dx \tag{1.1}$$

where E is the completion of the restriction on Ω of functions of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{a,b} = \left(\int_\Omega a(x)|\nabla u|^p\, dx + \int_{\partial\Omega} b(x)|u|^p\, d\sigma\right)^{\frac{1}{p}},$$

and X is a reflexive Banach space endowed with the norm

$$\|u\| = (\|u\|_{a,b}^p + \|u\|_{L^q(\Omega)}^p)^{\frac{1}{p}}.$$

Set $L^r(\Omega; g) := \{u : \Omega \to \mathbb{R} \mid u \text{ is Lebesgue measurable}, \int_\Omega g(x)|u|^r dx < \infty\}$. Under the assumptions (a),(b) and (g), we have the compact embedding $X \hookrightarrow L^r(\Omega; g)$ for $r \in (1, p^*)$, which will be stated in Section 2, so all integrals in (1.1) are well-defined and converge. Define the functional $J_\lambda : X \to \mathbb{R}$ by

$$J_\lambda(u) = \frac{1}{p}\int_\Omega a(x)|\nabla u|^p\, dx + \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p\, d\sigma + \frac{1}{q}\int_\Omega |u|^q\, dx - \frac{\lambda}{r}\int_\Omega g(x)|u|^r\, dx. \tag{1.2}$$

Then $J_\lambda \in C^1(X, \mathbb{R})$ with its derivative

$$\begin{aligned}\langle J_\lambda'(u), \phi\rangle = &\int_\Omega a(x)|\nabla u|^{p-2}\nabla u\nabla\phi\, dx + \int_{\partial\Omega} b(x)|u|^{p-2}u\phi\, d\sigma \\ &+ \int_\Omega |u|^{q-2}u\phi\, dx - \lambda\int_\Omega g(x)|u|^{r-2}u\phi\, dx, \quad \text{for all } \phi \in X.\end{aligned} \tag{1.3}$$

Therefore the critical points of J_λ correspond to the weak solutions of (P).

The main feature of problem (P) is that the quasilinear differential operator is affected by a different perturbation which behaves like $|u|^{q-2}u$, where $q \in (1, p^*)$. The relation of the power r with respect to the powers p and q affects the solvability of (P). In fact, this relation produces different competition between the different growths of the nonlinearities, which plays a crucial role in the study of geometrical feature of energy functional. In [17], the author studied (P) with Dirichlet boundary condition for $q = p$ and proved that (P) has a weak positive solution. Recently, in an interesting paper [18], under the same assumptions on a, b and g, the cases $p < r < q < p^*$ and $p < q < r < p^*$ with suitable parameter λ were treated and the existence of a weak solution for (P) was obtained via minimizing methods and mountain pass arguments. Motivated by the papers [18,17], we look for the existence of ground state and multiple solutions of (P) and related problem with more general nonlinearity via minimax arguments. We note that a ground state is a nontrivial solution which has the least energy among all the nontrivial solutions.

The purpose of this paper is two-fold. One purpose is to find multiple weak solutions to (P). Another purpose is to consider the critical case of (P), that is, the right side nonlinearity grows with critical exponent. In this case, the compactness of the embedding $X \hookrightarrow L^{p^*}(\Omega)$ fails, and it seems to be useless to impose hypothesis of (g) type for the critical term. To recover some sort of compactness, in spirit of [19], we consider a perturbation of the critical power. Namely, we study the problem

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) + |u|^{q-2}u = |u|^{p^*-2}u + \lambda g(x)|u|^{r-2}u, & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\partial_\nu u + b(x)|u|^{p-2}u = 0, & \text{on } \partial\Omega \end{cases} \tag{P*}$$

under the same assumptions on weight functions.

Now we state the main theorems in this paper. As to problem (P), we have the following results.

Theorem 1.1. *Let $1 < q < p < r < p^*$. Then for each $\lambda > 0$, problem (P) has a nonnegative ground state solution and infinitely many solutions in X.*

Theorem 1.2. *Let $p \leqslant q < r < p^*$. For each $\lambda > 0$, (P) has a positive ground state solution and a sequence of solutions u_k in X with $J_\lambda(u_k) \to \infty$ as $k \to \infty$.*