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On a phase-field system based on the Cattaneo law

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ABSTRACT

Our aim in this paper is to study a generalization of the Caginalp phase-field system based on the Maxwell–Cattaneo law for heat conduction and endowed with Neumann boundary conditions. In particular, we obtain well-posedness results and study the dissipativity of the associated solution operators. We also prove, when the enthalpy is conserved, the existence of the global attractor. We finally study the spatial behavior of solutions in a semi-infinite cylinder, assuming that such solutions exist and have a proper (spatial) decay at infinity.

1. Introduction

The Caginalp phase-field system

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta, \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t}, \tag{1.2}$$

where u is the order parameter, θ is the relative temperature and f is the derivative of a double-well potential F (here, we have set all physical parameters equal to 1), has been proposed in [1] to model phase transition phenomena, such as melting-solidification phenomena. This equation has been much studied; we refer the reader to, e.g., [2–16].

We can note that the generalized heat equation (1.2) is based on the usual Fourier law for heat conduction. Indeed, introducing the enthalpy

$$H = u + \theta, \tag{1.3}$$

one has

$$\frac{\partial H}{\partial t} = -\mathrm{div}\, q, \tag{1.4}$$

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where the thermal flux vector q is given by the usual Fourier law,

$$q = -\nabla\theta. \tag{1.5}$$

Now, one drawback of the Fourier law is that it predicts that thermal signals propagate with an infinite speed, which violates causality (see, e.g., [17]). Therefore, several alternative laws have been proposed and studied in [18–23]; one essential feature of these alternative models is that one ends up with a second-order (in time) equation for the temperature.

In particular, in the case of the Maxwell–Cattaneo law, one has

$$\left(1+\frac{\partial}{\partial t}\right)q = -\nabla\theta. \tag{1.6}$$

We then deduce from (1.4) that

$$\left(\frac{\partial}{\partial t}+\frac{\partial^2}{\partial t^2}\right)H = -\mathrm{div}\left(\left(1+\frac{\partial}{\partial t}\right)q\right), \tag{1.7}$$

hence the following equation for the temperature:

$$\frac{\partial^2\theta}{\partial t^2}+\frac{\partial\theta}{\partial t}-\Delta\theta = -\frac{\partial^2 u}{\partial t^2}-\frac{\partial u}{\partial t}. \tag{1.8}$$

Integrating (1.8) between 0 and t and setting

$$\alpha = \int_0^t \theta ds + \alpha_0 \quad \left(\theta = \frac{\partial\alpha}{\partial t}\right), \tag{1.9}$$

where α is called thermal displacement variable (here, α_0 is a priori fixed arbitrarily; see also Remark 2.1), we finally obtain

$$\frac{\partial^2\alpha}{\partial t^2}+\frac{\partial\alpha}{\partial t}-\Delta\alpha = -\frac{\partial u}{\partial t}-u+g, \tag{1.10}$$

where

$$g = \frac{\partial\theta}{\partial t}(0)+\theta(0)-\Delta\alpha_0+\frac{\partial u}{\partial t}(0)+u(0)\left(=\frac{\partial^2\alpha}{\partial t^2}(0)+\frac{\partial\alpha}{\partial t}(0)-\Delta\alpha_0+\frac{\partial u}{\partial t}(0)+u(0)\right). \tag{1.11}$$

Indeed, the term $\frac{\partial^2 u}{\partial t^2}$ in (1.8) would be very difficult to handle from a mathematical point of view. Note that (1.1) can be rewritten in the form

$$\frac{\partial u}{\partial t}-\Delta u+f(u) = \frac{\partial\alpha}{\partial t}. \tag{1.12}$$

Remark 1.1. Actually, the system still has an infinite propagation speed, due to the parabolic nature of (1.12). A fully hyperbolic model, i.e., one considers a hyperbolic relaxation of (1.12), is considered in [19].

Eqs. (1.10) and (1.12) have been studied in [20], in the case of Dirichlet boundary conditions. In particular, we studied the well-posedness, the stability, the dissipativity (when $g = 0$; note that g depends on the initial data) and the spatial behavior of solutions in a semi-infinite cylinder (assuming that such solutions exist).

Now, Neumann boundary conditions are also physically relevant. This has been considered in [18] (see also [19]), where the convergence of single trajectories to steady states has been studied. More precisely, there, a Neumann (on u) and a no flux (on q) boundary conditions have been considered. Note that this yields a Neumann boundary condition on θ, in view of (1.6). Furthermore, integrating (1.4) over the domain Ω occupied by the material, one obtains the conservation of the enthalpy, namely,

$$\frac{d}{dt}\int_\Omega Hdx = 0. \tag{1.13}$$

In this paper, we will actually directly take a Neumann boundary condition on θ (or, equivalently, on α). Note that this contains the above case, since it follows from (1.6) that

$$\left(1+\frac{\partial}{\partial t}\right)q.\nu = 0 \tag{1.14}$$

on the boundary, where ν is the unit outer normal vector, but is more general. Furthermore, one has, integrating (1.7) over Ω,

$$\left(1+\frac{\partial}{\partial t}\right)\int_\Omega Hdx = 0, \tag{1.15}$$

instead of the enthalpy conservation (1.13) (note that one has $H = u+\frac{\partial\alpha}{\partial t}$).