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On a class of abstract neutral functional differential equations

Eduardo Hernández [a,*], Michelle Pierri [a], Andréa Prokopczyk [b]

[a] Departamento de computação e matemática, Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto. Universidade de São Paulo, Ribeirão Preto, SP, Brazil

[b] Instituto de Biociencias, Letras e Ciencias Exatas, Depto de Matemática, Universidad Estudual Paulista, Rio Preto, SP, Brazil

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ABSTRACT

By using the theory of semigroups of growth α, we discuss the existence of mild solutions for a class of abstract neutral functional differential equations. A concrete application to partial neutral functional differential equations is considered.

1. Introduction

In this paper, we study the existence of mild solutions for a class of neutral functional differential equations of the form

$$\frac{d}{dt}[x(t)+g(t,x_t)] = Ax(t)+f(t,x_t), \quad t\in[0,a], \tag{1.1}$$

$$x_0=\varphi\in\Omega\subset\mathcal{B}, \tag{1.2}$$

where $A : D(A)\subset X\to X$ is an almost sectorial operator, $(X, \|\cdot\|)$ is a Banach space, $\mathcal{B}$ is the phase space ($\mathcal{B}=C([-r,0],X)$ or $\mathcal{B}=L^p([-r,0],X)$), $\Omega\subset\mathcal{B}$ is open and $g, f : [0,a]\times\Omega\to X$ are suitable functions.

There exists an extensive literature on abstract neutral differential equations treating the case in which A is a sectorial operator; see [1–6] and the references therein. Sectorial operators appear frequently in applications since many elliptic differential operators are sectorial when they are considered in Lebesgue spaces (L^p-spaces) or in spaces of continuous functions; see [7]. However, if we look at spaces of more regular functions such as the spaces of Hölder continuous functions, we find that these elliptic operators are not sectorial; see [7,8, Example 3.1.33]. Nevertheless, for these operators estimates such as

$$\|(\lambda-A)^{-1}\|\le\frac{M}{|\lambda|^{1-\alpha}}, \quad \lambda\in\Sigma_{\omega,\theta}=\{\lambda\in\mathbb{C} : |\arg(\lambda-\omega)|<\theta\}, \tag{1.3}$$

* Corresponding author. Tel.: +55 1636023670; fax: +55 163633-2660.
E-mail addresses: lalohm@icmc.usp.br (E. Hernández), michellepierri@yahoo.com.br (M. Pierri), andrea.prokopczyk@gmail.com (A. Prokopczyk).

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with $\alpha \in (0,1)$, $\omega \in \mathbb{R}$ and $\theta \in \left(\frac{\pi}{2}, \pi\right)$ are available (see [8]), which allow one to define an associated "semigroup" by means of the Dunford integral

$$T(t) = \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}(\lambda - A)^{-1}d\lambda, \quad t > 0, \ (\Gamma_\theta = \{te^{i\theta} : t \in \mathbb{R}\setminus\{0\}\}). \tag{1.4}$$

Under the above conditions, the operator A is called almost sectorial and the operator family $\{T(t), T(0) = I : t \geq 0\}$ is said the semigroup of growth α generated by A. The semigroup $(T(t))_{t\geq 0}$ has properties similar at those of analytic semigroup which allows to study some classes of partial differential equations via the useful methods of semigroup theory. Concerning almost sectorial operators, semigroup of growth α and applications to partial differential equations, we refer the reader to [8–14] and the references therein.

The literature on ordinary neutral differential equations is quite complete, and related this matter we refer the reader to the book by Hale [15]. Concerning abstract neutral differential equations similar to (1.1)–(1.2), we cite Ezzinbi et al. [16], Datko [1], Hernández and O'Regan [2,17,18] and Hernández and Balachandran [3] for equations with finite delay and Hernández et al. [4,5,19] for neutral equations with unbounded delay. To the best of our knowledge, the problem of the existence of solutions for abstract neutral differential equations involving almost sectorial operators is an untreated topic in the literature. This fact, is the main motivation of this paper.

Neutral differential equations arise in many areas of applied mathematics, and for this reason, those equations have been of a great interest during the last few decades. Partial neutral differential equations arise, for instance, in the theory of heat conduction in fading memory material. In the classical theory of heat conduction, it is assumed that the internal energy and the heat flux depends linearly on the temperature u and on its gradient ∇u. Under these conditions, the classical heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [20,21], the internal energy and the heat flux are described as functionals of u and u_x. The next system, see [22–25], has been frequently used to describe this phenomena,

$$\frac{d}{dt}\left[u(t,x) + \int_{-\infty}^{t} k_1(t-s)u(s,x)ds\right] = c\Delta u(t,x) + \int_{-\infty}^{t} k_2(t-s)\Delta u(s,x)ds,$$
$$u(t,x) = 0, \quad x \in \partial\Omega.$$

In this system, $\Omega \subset \mathbb{R}^n$ is open, bounded and has smooth boundary, $(t,x) \in [0,\infty)\times\Omega$, $u(t,x)$ represents the temperature in x at the time t, c is a physical constant and $k_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are the internal energy and the heat flux relaxation respectively. By assuming the solution $u(\cdot)$ is known on $(-\infty, 0]$, $k_1 \equiv 0$ on (r,∞) and $k_2 \equiv 0$, we can transform this system into the abstract form (1.1)–(1.2).

Next, we introduce some specific notations that will be used throughout the text. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. In this paper, $\mathcal{L}(Z, W)$ represents the space of bounded linear operators from Z into W endowed with norm of operators denoted $\|\cdot\|_{\mathcal{L}(Z,W)}$, and we write $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ when $Z = W$. In addition, $B_l(z, Z)$ denotes the closed ball with center at z and radius l in Z. For a closed linear operator $S : D(S) \subset Z \to Z$, we denote by $[S]$ the domain of S endowed with the graph norm $\|x\|_{[S]} = \|x\|_Z + \|Sx\|_Z$ and by $\rho(S)$ the resolvent set of S. As usual, for an interval $I \subset \mathbb{R}$, the notation $C(I, Z)$ represents the space formed for all the bounded continuous functions from I into Z endowed with the sup-norm denoted by $\|\cdot\|_{C(I,Z)}$. In addition, $L^p(I, X)$, $p \geq 1$, denotes the space formed for all the classes of Lebesgue-integrable functions from I into X endowed with the norm $\|h\|_{L^p(I,X)} = \left(\int_I \|h(s)\|^p ds\right)^{\frac{1}{p}}$.

Throughout this paper, $(X, \|\cdot\|)$ is a Banach space, $A : D(A) \subset X \to X$ is an almost sectorial operator and $(T(t))_{t\geq 0}$ is the semigroup of growth α generated by A. For simplicity, next we assume $\omega = 0$.

The next lemma consider some properties of the semigroup $(T(t))_{t\geq 0}$. The proof follows arguing as in [7, Chapter II].

Lemma 1.1. *Under the above conditions, the followings properties are verified.*

(a) *The operator A is closed, $T(t+s) = T(t)T(s)$ and $AT(t)x = T(t)Ax$ for all $t, s \in [0,\infty)$ and each $x \in D(A)$.*
(b) *$T(\cdot) \in C^1((0,\infty), \mathcal{L}(X))$ and $\frac{d}{dt}T(t) = AT(t)$ for all $t > 0$.*
(c) *For all $\varepsilon > 0$ and every $n \in \mathbb{N}\cup\{0\}$, $A^nT(\cdot) \in C((0,\infty), X)$ and there exists $D_{n,\varepsilon} > 0$ such that $\|A^nT(t)\|_{\mathcal{L}(X)} \leq \frac{D_{n,\varepsilon}e^{(\omega+\varepsilon)t}}{t^{n+\alpha}}$ for all $t > 0$.*

Next, we include some remarks on the abstract Cauchy problem

$$x'(t) = Ax(t) + \xi(t), \quad t \in [0, a], \tag{1.5}$$
$$x(0) = x \in X, \tag{1.6}$$

where $\xi \in L^q([0,a], X)$ and $q > \frac{1}{1-\alpha}$. From [10,12], we note the followings concepts of solution for (1.5)–(1.6).

Definition 1.1. A function $u : [0,b] \to X$, $0 < b \leq a$, is said a mild solution of (1.5)–(1.6) on $[0,b]$ if

$$u(t) = T(t)x + \int_0^t T(t-s)\xi(s)ds, \quad \forall t \in [0,b]. \tag{1.7}$$